GROUPOID CROSSED PRODUCTS

A Thesis

Submitted to the Faculty

in partial fulfillment of the requirements for the

degree of

Doctor of Philosophy

in

Mathematics

by

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DARTMOUTH COLLEGE

Hanover, New Hampshire

May 29, 2009

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Abstract

We present a number of findings concerning groupoid dynamical systems and groupoid crossed products. The primary result is an identification of the spectrum of the groupoid crossed product when the groupoid has continuously varying abelian stabilizers and a well behaved orbit space. In this case, the spectrum of the crossed product is homeomorphic, via an induction map, to a quotient of the spectrum of the crossed product by the stabilizer group bundle. The main theorem is also generalized in the groupoid algebra case to an identification of the primitive ideal space. This generalization replaces the assumption that the orbit space is well behaved with an amenability hypothesis. We then use induction to show that the primitive ideal space of the groupoid algebra is homeomorphic to a quotient of the dual of the stabilizer group bundle. In both cases the identification is topological. We then apply these theorems in a number of examples, and examine when a groupoid algebra has Hausdorff spectrum. As a separate result, we also develop a theory of principal groupoid group bundles and locally unitary groupoid actions. We prove that such actions are characterized, up to exterior equivalence, by a cohomology class which arises from a principal bundle. Furthermore, we also demonstrate how to construct a locally unitary action from a given principal bundle. This last result uses a duality theorem for abelian group bundles which is also included as part of this thesis.

Preface

This thesis contains a number of results concerning groupoid crossed products and groupoid C^*-algebras which have been developed through the author's graduate studies at Dartmouth College. Groupoid crossed products inherit the generality of groupoids. In particular, they simultaneously generalize group crossed products, transformation group algebras, and groupoid algebras.

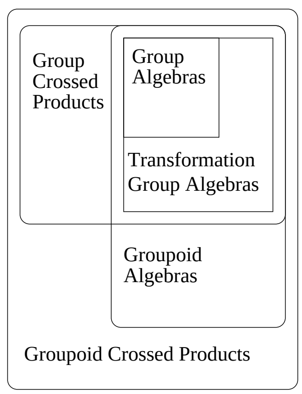

Figure 1: Groupoid crossed products are very general objects.

Before we describe the structure of the thesis we should give some idea of what the prerequisites are for understanding its contents. The reader who is familiar with groupoids and groupoid crossed products will encounter no difficulties. Because the field is so new, the author has taken some trouble to keep the presentation as self-contained as possible. Someone with a basic knowledge of C^*-algebras and functional analysis could expect to understand much of this thesis, particularly the first two chapters, but may eventually run into trouble. A reasonable set of required reading is the following list of references:

- *An Invitation to C^*-algebras*, William Arveson, Chapter 1, [Arv76]
- *Morita Equivalence and Continuous-Trace C^*-algebras*, Iain Raeburn and Dana P. Williams, Chapters 1, 2, 3, Sections 4.1, 4.2, and Appendix A, [RW98]

- *Crossed Products of C^*-algebras*, Dana P. Williams, Chapters 1, 2 and Appendix C. [Wil07]

These references come highly recommended by the author and are each worth reading. In particular, we will be citing these books frequently. Some other works that we will cite frequently are:

- *A Groupoid Approach to C^*-algebras*, Jean Renault, [Ren80]
- *Coordinates In Operator Algebras*, Paul Muhly, [Muh]
- *Continuous-Trace Groupoid C^*-algebras III*, Paul Muhly, Jean Renault, and Dana Williams. [PSMW96].
- *Renault's Equivalence Theorem for Groupoid Crossed Products*, Paul Muhly and Dana P. Williams, [MW08]
- *The Ideal Structure of Groupoid Crossed Product C^*-algebras*, Jean Renault, [Ren91]

Those readers interested in chasing down citations will find that much of the groupoid theory in this work is inspired by the first three references listed above and much of the crossed product theory is inspired by the last two. What's more, readers are encouraged to look up references. There has been some effort made to cite results as they appear in their original context, or if that is not possible, to include a remark which explains how to extract the given statement from the statement in the reference.

As for the structure of this thesis, because groupoid crossed products rely heavily on groupoid theory, and because groupoid theory is a relatively new field in and of itself, we begin with an introduction to groupoid basics in Chapter 1. This includes definitions and elementary properties of groupoids in Section 1.1 and actions of groupoids on topological spaces in Section 1.2. Also included in this chapter are more advanced results concerning groupoid equivalence in Section 1.2.1 and groupoid amenability in Section 1.3. Next, in Chapter 2 we explore the structure of groupoid group bundles. In Section 2.1 we develop the notion of a principal S-bundle and show that they are characterized, up to isomorphism, by an associated cohomology class. In Section 2.2 we demonstrate a generalization of Pontryagin duality for abelian, continuously varying group bundles. In Section 2.3 we describe a counterexample which, in addition to being interesting in its own right, shows that the work done in Section 2.2 is necessary. In Chapter 3 we introduce the basics of groupoid dynamical systems and groupoid crossed products. We start by giving a brief overview of upper-semicontinuous bundle theory in Section 3.1 and then define a groupoid dynamical system in Section 3.2. In Section 3.3 we develop the theory of covariant representations of groupoid dynamical systems. We then use these covariant representations in

Section 3.4 to define the groupoid crossed product. In this section we also introduce Renault's Disintegration Theorem, which will be an important tool. In Chapter 4 we describe a number of special cases of groupoid crossed products, and show that groupoid crossed products generalize groupoid algebras and group crossed products as in Figure 1. Not only does this connect the theory to existing mathematics, but these constructions will prove essential in later chapters. A modest result, that is nonetheless interesting, is a generalization of the Stone-von Neumann theorem to groupoids, presented in Section 4.4.1. Next, in Chapter 5 we present some useful and interesting properties of groupoid crossed products. In Sections 5.1 and 5.2 we mainly deal with technical results. In particular, since it is the first really high level portion of the text, Section 5.1 contains restatements of quite a few theorems which are too complicated to prove here. On the other hand, this section also presents results concerning transitive groupoid crossed products which, while basic, are new. In Section 5.3 and 5.4 we define the notion of unitary and locally unitary actions. In particular, we show that for unitary actions the crossed product reduces to a tensor product. For locally unitary actions the results are more interesting. We show that these actions are characterized, up to exterior equivalence, by a principal bundle. We also show that any principal bundle can be used to construct a locally unitary action. Moving on, Chapter 6 contains the primary results of the thesis. Section 6.1 describes a technique for inducing representations from a closed subgroupoid up to the whole crossed product. We will eventually this induction technique to identify the spectrum of certain crossed product algebras. In Section 6.2 we show, as long as the orbit space of the groupoid is T_0, that every irreducible representation of the crossed product is equivalent to the induction of an irreducible representation of a fibre. Then in Section 6.3 we show, whenever the stabilizers are abelian and continuously varying and the orbit space is T_0, that this induction map factors to a homeomorphism of a quotient of the spectrum of $A \rtimes S$ onto the spectrum of $A \rtimes G$. In Chapter 7 we apply these results to various examples and special cases. In particular, Section 7.1 contains a strengthening of the results of Section 6.3 in the groupoid C^*-algebra case. Section 7.2 describes how these results can be applied to transformation groupoids. Moreover, examples are given and the theory is connected back to similar results for transformation group algebras. The last portion of the thesis is Section 7.3, which contains an analysis of when groupoid C^*-algebras have Hausdorff spectrum. While no conclusive answer is given, this section has some intriguing constructions as well as several nice counterexamples.

We have outlined the logical structure of the thesis in the following diagram. In particular, those readers interested only in induction and the fine structure result may skip the branch containing Chapter 2 while whose readers only interested in locally unitary actions can ignore Chapters 6 and 7.

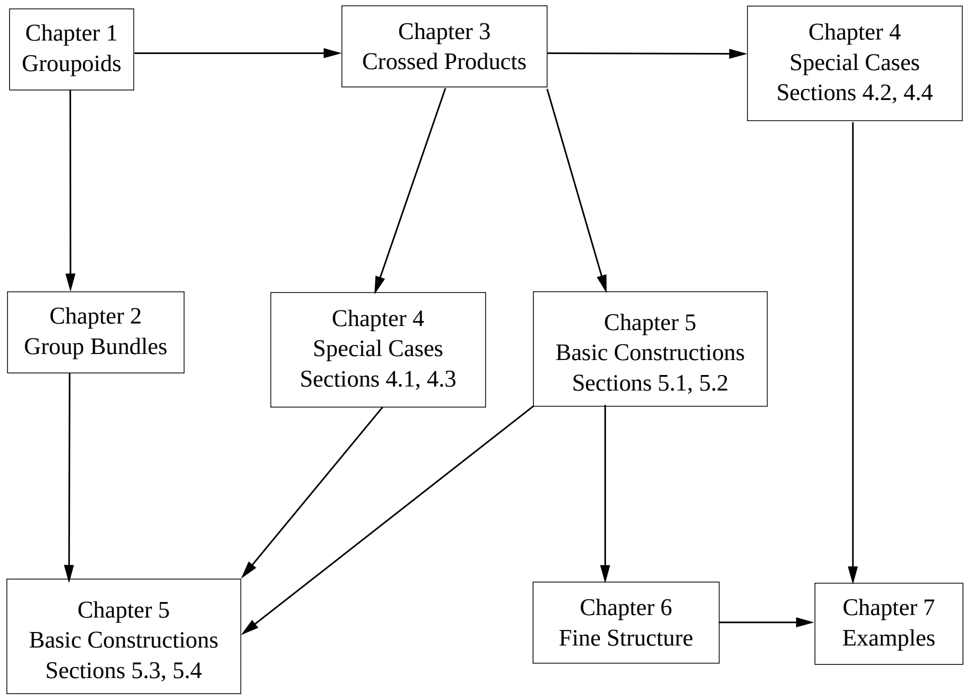

Figure 2: The logical structure of the thesis.

Acknowledgments

First and foremost I would like to thank my adviser, Dana Williams. He has answered all of my incessant questions patiently ever since I got here five years ago. I would also like to thank the host of other people, including Jonathan Brown and Paul Muhly, who have helped me with proofs, supplied clever arguments, or just made their knowledge available. Lastly, I would like to thank Naomi for being awesome.

Contents

Chapter 1

Groupoids

In this chapter we present an overview of basic groupoid theory. In Section 1.1 we define what a groupoid is and outline some of the elementary facts and notation. We also discuss the notion of a stabilizer groupoid and an orbit groupoid. The stabilizer groupoid will play an important role through out. Section 1.2 contains the basic constructions concerning groupoid actions. It is notable that we will separate the condition that the structure map be open from the usual definition of a groupoid action, see Remark 1.61. The remainder of the section is basically review. We define groupoid equivalence and give a construction of the imprimitivity groupoid and then in Section 1.3 we give the briefest description of groupoid amenability.

1.1 Groupoid Basics

Groupoids are essentially groups with a partially defined multiplication. While it may not seem like much, this has a tremendous impact on their structure. This section will introduce some of the basic properties of groupoids, but we must start with their definition. The following draws heavily from [Muh].

Definition 1.1. Suppose G is a set and $G^{(2)} \subseteq G \times G$. Then G is a *groupoid* if there are maps $(\gamma, \eta) \mapsto \gamma\eta$ from $G^{(2)}$ into G and $\gamma \mapsto \gamma^{-1}$ from G into G such that:

(a) *(associativity)* If (γ, η) and (η, ξ) are in $G^{(2)}$ then so are $(\gamma\eta, \xi)$ and $(\gamma, \eta\xi)$, and we have $(\gamma\eta)\xi = \gamma(\eta\xi)$.

(b) *(involution)* For all $\gamma \in G$ we have $(\gamma^{-1})^{-1} = \gamma$.

(c) *(cancellation)* For all $\gamma \in G$ we have $(\gamma^{-1}, \gamma) \in G^{(2)}$ and if $(\gamma, \eta) \in G^{(2)}$ then $\gamma^{-1}(\gamma\eta) = \eta$ and similarly $(\gamma\eta)\eta^{-1} = \gamma$.

The set $G^{(2)}$ is called the set of *composable pairs*, when $(\gamma, \eta) \in G^{(2)}$ we say γ and η are *composable*, and γ^{-1} is called the *inverse* of γ.

Some of the formulas in Definition 1.1 are reminiscent of the usual group axioms. However, unlike the group case, the partially defined multiplication implies that many different elements of G act like units.

Definition 1.2. Suppose G is a groupoid. Then the set of elements of G such that $\gamma = \gamma^{-1} = \gamma^2$ is denoted $G^{(0)}$ and is called the *unit space*. The map $r : G \to G^{(0)}$ such that $r(\gamma) = \gamma\gamma^{-1}$ is called the *range map* and the map $s : G \to G^{(0)}$ such that $s(\gamma) = \gamma^{-1}\gamma$ is called the *source map*. Given $u \in G^{(0)}$ we will use the notation $G_u := s^{-1}(u)$ and $G^u := r^{-1}(u)$.

Remark 1.3. Suppose G is a groupoid with A and B subsets of G. We will use the notation

$$AB = A \cdot B := \{\gamma\eta : \gamma \in A, \eta \in B, (\gamma, \eta) \in G^{(2)}\}, \qquad A^{-1} := \{\gamma^{-1} : \gamma \in A\}.$$

It's important to realize that AB may be badly behaved. For instance, AB may not contain either A or B, and is actually empty if $A \times B \cap G^{(2)} = \emptyset$.

Of course, given any new class of objects, there is also a new class of homomorphisms.

Definition 1.4. Suppose G and H are groupoids. A map $\phi : G \to H$ is a *groupoid homomorphism* if and only if whenever $(\gamma, \eta) \in G^{(2)}$ then $(\phi(\gamma), \phi(\eta)) \in H^{(2)}$ and in this case $\phi(\gamma\eta) = \phi(\gamma)\phi(\eta)$. If ϕ is also bijective then its called a *groupoid isomorphism*.

The next proposition outlines some of the basic properties of the range map, source map, and the elements of $G^{(0)}$.

Proposition 1.5. *Suppose G is a groupoid.*

(a) *Given $\gamma, \eta \in G$ we have $(\gamma, \eta) \in G^{(2)}$ if and only if $s(\gamma) = r(\eta)$.*

(b) *If $(\gamma, \eta) \in G^{(2)}$ then $r(\gamma\eta) = r(\gamma)$ and $s(\gamma\eta) = s(\eta)$.*

(c) *If $\gamma \in G$ then $r(\gamma) = s(\gamma^{-1})$ and $s(\gamma) = r(\gamma^{-1})$.*

(d) *If $(\gamma, \eta) \in G^{(2)}$ then $(\eta^{-1}, \gamma^{-1}) \in G^{(2)}$ and $(\gamma\eta)^{-1} = \eta^{-1}\gamma^{-1}$.*

(e) *If $\gamma \in G$ then $r(\gamma), s(\gamma) \in G^{(0)}$. Furthermore, r and s are retractions onto $G^{(0)}$.*[1]

(f) *If $\gamma \in G$ then $(r(\gamma), \gamma), (\gamma, s(\gamma)) \in G^{(2)}$, $r(\gamma)\gamma = \gamma$, and $\gamma s(\gamma) = \gamma$.*

[1]Given a set X and a subset A of X a map $f : X \to A$ is a retraction if f restricted to A is the identity.

Proof. Part **(a)**: Suppose $(\gamma, \eta) \in G^{(2)}$. We know from the cancellation condition of Definition 1.1 that $(\gamma^{-1}, \gamma) \in G^{(2)}$. Using associativity we have $(\gamma^{-1}\gamma)\eta = \gamma^{-1}(\gamma\eta)$ which, after applying cancellation, gives us $(\gamma^{-1}\gamma)\eta = \eta$. Using the first part of cancellation again, along with involution, we get $(\eta, \eta^{-1}) \in G^{(2)}$, allowing us to multiply the previous equality by η. This yields

$$((\gamma^{-1}\gamma)\eta)\eta^{-1} = \eta\eta^{-1}.$$

Finally, using cancellation once again, we conclude

$$s(\gamma) = \gamma^{-1}\gamma = \eta\eta^{-1} = r(\eta).$$

Next, suppose $s(\gamma) = r(\eta)$. Once more, condition (c) of Definition 1.1 tells us that $(\gamma^{-1}, \gamma) \in G^{(2)}$, and if we use involution we can similarly conclude that $(\gamma, \gamma^{-1}) \in G^{(2)}$. Associativity implies that $(\gamma, \gamma^{-1}\gamma) \in G^{(2)}$. Since $\gamma^{-1}\gamma = s(\gamma) = r(\eta) = \eta\eta^{-1}$ it follows that $(\gamma, \eta\eta^{-1}) \in G^{(2)}$. Next, using Definition 1.1 on η in a similar fashion, we have $(\eta\eta^{-1}, \eta) \in G^{(2)}$ and therefore, by associativity, $(\gamma, (\eta\eta^{-1})\eta) \in G^{(2)}$. However, cancellation implies that $(\eta\eta^{-1})\eta = \eta$ so that $(\gamma, \eta) \in G^{(2)}$.

Part **(c)**: Suppose $\gamma \in G$. Then using involution

$$r(\gamma^{-1}) = \gamma^{-1}(\gamma^{-1})^{-1} = \gamma^{-1}\gamma = s(\gamma).$$

The calculation that $s(\gamma^{-1}) = r(\gamma)$ is similar.

Part **(b)**: Suppose $(\gamma, \eta) \in G^{(2)}$. Then $((\gamma\eta)^{-1}, \gamma\eta) \in G^{(2)}$, as well as (η, η^{-1}). Applying associativity to (γ, η) and (η, η^{-1}) gives us $(\gamma\eta, \eta^{-1}) \in G^{(2)}$. Applying associativity again to $(\gamma\eta, \eta^{-1})$ and $((\gamma\eta)^{-1}, \gamma\eta)$ implies $((\gamma\eta)^{-1}, (\gamma\eta)\eta^{-1}) \in G^{(2)}$. Using cancellation we conclude that $((\gamma\eta)^{-1}, \gamma) \in G^{(2)}$. It follows from (c) and (a) that

$$r(\gamma\eta) = s((\gamma\eta)^{-1}) = r(\gamma).$$

The calculation which shows $s(\gamma\eta) = s(\eta)$ is similar.

Part **(d)**: Suppose $(\gamma, \eta) \in G^{(2)}$. Then $s(\gamma) = r(\eta)$ and using part (c) we have $r(\gamma^{-1}) = s(\eta^{-1})$ so that $(\eta^{-1}, \gamma^{-1}) \in G^{(2)}$. Applying cancellation we get

$$\eta^{-1}\gamma^{-1} = ((\eta^{-1}\gamma^{-1})(\gamma\eta))(\gamma\eta)^{-1}.$$

However, applying associativity and cancellation, we have

$$\begin{aligned} \eta^{-1}\gamma^{-1} &= (\eta^{-1}\gamma^{-1}(\gamma\eta))(\gamma\eta)^{-1} \\ &= (\eta^{-1}(\gamma^{-1}(\gamma\eta)))(\gamma\eta)^{-1} \\ &= (\eta^{-1}\eta)(\gamma\eta)^{-1}. \end{aligned}$$

Technically, we have to know that η and $(\gamma\eta)^{-1}$ are composable before we can apply associativity. However part (b) implies $s(\eta) = s(\gamma\eta) = r((\gamma\eta)^{-1})$ so that $(\eta, (\gamma\eta)^{-1}) \in G^{(2)}$. Now we can use associativity and conclude

$$\begin{aligned}\eta^{-1}\gamma^{-1} &= (\eta^{-1}\eta)(\gamma\eta)^{-1} \\ &= \eta^{-1}(\eta(\gamma\eta)^{-1}) = (\gamma\eta)^{-1}.\end{aligned}$$

Part **(e)**: If $\gamma \in G$ then $s(\gamma)^{-1} = (\gamma^{-1}\gamma)^{-1} = \gamma^{-1}\gamma = s(\gamma)$ by involution and part (d). Since $s(\gamma)^{-1} = s(\gamma)$ we conclude from part (c) that the range and source of $s(\gamma)$ are equal and therefore $(s(\gamma), s(\gamma)) \in G^{(2)}$. Finally,

$$s(\gamma)s(\gamma) = \gamma^{-1}\gamma\gamma^{-1}\gamma = \gamma^{-1}\gamma = s(\gamma)$$

by cancellation. Thus $s(\gamma) \in G^{(0)}$. Since $r(\gamma) = s(\gamma^{-1})$ this also shows $r(\gamma) \in G^{(0)}$. Next, if $u \in G^{(0)}$ then $s(u) = u^{-1}u = u^2 = u$ and $r(u) = uu^{-1} = u^2 = u$. Hence r and s are retractions onto $G^{(0)}$.

Part **(f)**: Suppose $\gamma \in G$. Then $s(r(\gamma)) = s(\gamma\gamma^{-1}) = r(\gamma)$ by parts (b) and (c). Thus $r(\gamma)$ and γ are composable and by cancellation

$$r(\gamma)\gamma = \gamma\gamma^{-1}\gamma = \gamma.$$

The proof that γ and $s(\gamma)$ are composable and that $\gamma s(\gamma) = \gamma$ is similar. $\square$

Remark 1.6. Properties (a) - (b) of Proposition 1.5 help explain the terminology behind the range and source maps and why groupoid elements are sometimes called arrows. Colloquially, every arrow in a groupoid has a range and a source given by r and s. Arrows are composable if and only if the range of one matches the source of the second, and the range and source of the composition are exactly what you would expect them to be. Properties (c) - (d) show that the inverse of an arrow goes in the "opposite direction" and that composition and inverses get along nicely. Property (f) explains why elements of $G^{(0)}$ are called units; they act like identities on elements with which they are composable.

An alternative source for intuition regarding the partially defined multiplication, and another reason for the arrow terminology, is to think of a groupoid as a (small) category where every morphism is an isomorphism. This is actually equivalent to Definition 1.1.

We can also describe some basic properties of groupoid homomorphisms.

Proposition 1.7. *Suppose G and H are groupoids and $\phi : G \to H$ is a groupoid homomorphism.*

(a) *Given $u \in G^{(0)}$ we have $\phi(u) \in H^{(0)}$.*

(b) *Given* $\gamma \in G$ *we have* $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$.

(c) *For all* $\gamma \in G$ *we have* $r(\phi(\gamma)) = \phi(r(\gamma))$ *and* $s(\phi(\gamma)) = \phi(s(\gamma))$.

Proof. Part **(a)**: Suppose $u \in G^{(0)}$. Then

$$\phi(u) = \phi(u^2) = \phi(u)\phi(u).$$

Composing both sides with $\phi(u)^{-1}$ and using cancellation yields

$$\phi(u)\phi(u)^{-1} = (\phi(u)\phi(u))\phi(u)^{-1} = \phi(u).$$

It follows that $\phi(u) = r(\phi(u)) \in H^{(0)}$.

Part **(b)**: Suppose $\gamma \in G$. Since γ and γ^{-1} are composable we have $\phi(\gamma\gamma^{-1}) = \phi(\gamma)\phi(\gamma^{-1})$. Next, we can compose both sides with $\phi(\gamma)^{-1}$ and use cancellation to obtain

$$\phi(\gamma)^{-1}\phi(\gamma\gamma^{-1}) = \phi(\gamma)^{-1}\phi(\gamma)\phi(\gamma^{-1}) = \phi(\gamma^{-1}).$$

However $\gamma\gamma^{-1} = r(\gamma) \in G^{(0)}$. By part (a) we know $\phi(r(\gamma)) \in H^{(0)}$ and using Proposition 1.5 to view $\phi(r(\gamma))$ as a right identity we have

$$\phi(\gamma)^{-1} = \phi(\gamma^{-1}).$$

Part **(c)**: Using part (b), we have

$$\phi(r(\gamma)) = \phi(\gamma\gamma^{-1}) = \phi(\gamma)\phi(\gamma)^{-1} = r(\phi(\gamma)).$$

The proof for the source map is exactly the same. □

In order to do any interesting functional analysis using groupoids you have to assume that there is a topology floating around, or at least a Borel structure.

Definition 1.8. Suppose G is a groupoid with a topology and $G^{(2)}$ is endowed with the relative product topology. Then G is a *topological groupoid* if the maps $(\gamma, \eta) \mapsto \gamma\eta$ from $G^{(2)}$ to G and $\gamma \mapsto \gamma^{-1}$ from G to G are continuous. If G has a Borel structure such that $G^{(2)}$ is a Borel subset of $G \times G$ and the above maps are Borel then we call G a *Borel groupoid*. Furthermore, if G is a topological groupoid then we view G as a Borel groupoid with the Borel structure coming from the topology, after we give $G^{(2)}$ the relative product Borel structure.

Remark 1.9. Almost without exception we will only be interested in topological groupoids where the topology is locally compact Hausdorff. Oftentimes, we will also assume that the topology is second countable.

Proposition 1.10. *If* G *is a locally compact Hausdorff groupoid then*

(a) *the range and source maps are continuous,*

(b) *the unit space $G^{(0)}$ is closed in G, and*

(c) *the set $G^{(2)}$ is closed in $G \times G$.*

Proof. It is clear that r and s are continuous since the composition and inversion operations are continuous. Now, suppose $\{u_i\} \in G$ is a net and $u_i \to u$. Using the fact that r is continuous we have $r(u_i) \to r(u)$. However, r is a retraction onto $G^{(0)}$ by Proposition 1.5 so $r(u_i) = u_i$. It follows that $u_i \to r(u)$. Since G is Hausdorff $u = r(u) \in G^{(0)}$ and $G^{(0)}$ is closed. Finally, suppose $(\gamma_i, \eta_i) \in G^{(2)}$ and $(\gamma_i, \eta_i) \to (\gamma, \eta)$. Then we have $\gamma_i \to \gamma$ and $\eta_i \to \eta$. It follows that $s(\gamma_i) \to s(\gamma)$ and $r(\eta_i) \to r(\eta)$. However, $s(\gamma_i) = r(\eta_i)$ for all i and G is Hausdorff so $s(\gamma) = r(\eta)$ and $(\gamma, \eta) \in G^{(2)}$. □

An important class of groupoids are those for which the unit space is also open. We will see later that they are very rigid objects with some nice properties.

Definition 1.11. Suppose G is a locally compact Hausdorff groupoid. If $G^{(0)}$ is open in G then we say that G is an *r-discrete* groupoid.

Remark 1.12. We are using the older definition of r-discrete as given in [Ren80]. However, this definition has fallen out of favor. Currently r-discrete groupoids are those for which the unit space is open and the range map is a local homeomorphism. These groupoids are also called etalé groupoids. We will see in Proposition 1.29 that this is equivalent to assuming that the groupoid is r-discrete, in the classical sense, and has a Haar system.

Groupoids are very general objects and extend a number of well understood structures. The following examples show how groupoids generalize groups, sets, equivalence relations, and transformation groups as in Figure 1.1.

Example 1.13. Suppose H is a locally compact Hausdorff group. If we let $H^{(2)} = H \times H$ and give H its group operations then H is a locally compact Hausdorff groupoid. In this case $r(g) = s(g) = e$ for all $g \in H$, where e is the identity of H.

Example 1.14. Suppose X is a locally compact Hausdorff space. If we let $X^{(2)}$ be the diagonal in $X \times X$ then, with the trivial operations $(x, x) \mapsto x$ and $x^{-1} \mapsto x$, it is easy to see that X is a locally compact Hausdorff groupoid. In this case $X^{(0)} = X$, every element is a unit, and X is known as a "cotrivial" groupoid.

Example 1.15. Let X be a locally compact Hausdorff space. Suppose $R \subset X \times X$ is locally compact Hausdorff in the relative topology and defines an equivalence relation on X by $x \sim y$ if and only if $(x, y) \in R$. We let

$$R^{(2)} = \{((x, y), (w, z)) \in R \times R : y = w\}$$

and define

$$(x,y)(y,z) := (x,z), \qquad\qquad (x,y)^{-1} := (y,x).$$

With these operations R is a locally compact Hausdorff groupoid. The unit space is $R^{(0)} = \{(x,x) : x \in X\}$, and we usually make the obvious identification of $R^{(0)}$ with X. Under this identification $r(x,y) = x$ and $s(x,y) = y$. If $R = X \times X$ then R is called the "trivial" groupoid. If $R = R^{(0)} \cong X$ then X is the "cotrivial" groupoid from Example 1.14

Example 1.16. Suppose H is a locally compact Hausdorff group acting on a locally compact Hausdorff space X. Let $G = H \times X$ and

$$G^{(2)} = \{((g,x),(h,y)) \in G \times G : y = g^{-1} \cdot x\}.$$

Given $((g,x),(h,y)) \in G^{(2)}$ we define

$$(g,x)(h,y) := (gh,x), \qquad\qquad (g,x)^{-1} := (g^{-1}, g^{-1} \cdot x).$$

It's not hard to see that with these operations G is a locally compact Hausdorff groupoid. The range and source maps are

$$\begin{aligned} s(g,x) &= (g^{-1}, g^{-1} \cdot x)(g,x) = (g^{-1}g, g^{-1} \cdot x) = (e, g^{-1} \cdot x), \quad \text{and} \\ r(g,x) &= (g,x)(g^{-1}, g^{-1} \cdot x) = (gg^{-1}, x) = (e,x) \end{aligned}$$

where e is the unit of H. In this case $G^{(0)} = \{(e,x) : x \in X\}$ and we will usually identify $G^{(0)}$ with X. Under this identification $s(g,x) = g^{-1} \cdot x$ and $r(g,x) = x$. This type of groupoid is called a "transformation group groupoid" or just a "transformation groupoid" for short. Transformation group groupoids generalize group actions in the

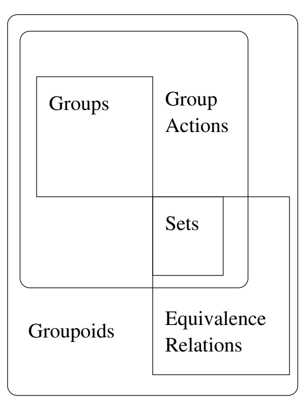

Figure 1.1: Groupoids generalize many different objects.

sense that the group action is completely determined by the associated transformation groupoid.

Remark 1.17. Suppose $G = H \times X$ is a transformation groupoid. If H is discrete then X is open in G and G is r-discrete. Similarly if X is open in G then the identity must be open (as a singleton) in H, and therefore H is discrete. Thus r-discrete transformation groupoids correspond to discrete group actions. One of the reasons that r-discrete groupoids are important is because they generalize discrete group actions in this way.

Examples 1.13 through 1.15 are all slightly degenerate in some sense. Example 1.16 describes a class of groupoids which is much more general. In fact, much of the inspiration for groupoids can be traced back to the transformation group case.

Remark 1.18. In Examples 1.15 and 1.16 we were able to identify the unit space of G with an associated space not contained in G. This kind of identification happens fairly frequently, and we will often treat $G^{(0)}$ as if it exists "outside" G.

There are also many groupoids which do not come from transformation groups, equivalence relations, or one of the examples presented above.

Example 1.19. Suppose X is a *compact* Hausdorff space and $\sigma : X \to X$ is a covering map. Let

$$G = \{(x, n, y) \in X \times \mathbb{Z} \times X : \exists\, k, l \geq 0 \text{ s.t. } n = l - k, \sigma^k x = \sigma^l y\}.$$

Then define

$$G^{(2)} = \{((x, n, y), (w, m, z)) \in G \times G : y = w\}$$

and give G the operations

$$(x, n, y)(y, m, z) := (x, n + m, z), \qquad\qquad (x, n, y)^{-1} := (y, -n, x).$$

With these operations G is a groupoid with unit space $G^{(0)} = \{(x, 0, x) \in G : x \in X\}$. We usually make the obvious identification of $G^{(0)}$ with X. Under this identification $r(x, n, y) = x$ and $s(x, n, y) = y$. Furthermore, in these circumstances G carries a topology making it into a locally compact Hausdorff r-discrete groupoid [Dea95, Theorem 1]. This is known as the "Deaconu-Renault groupoid" associated to (X, σ).

Example 1.20. Suppose $E = (E^0, E^1, r, s)$ is a row-finite[2] directed graph without sources. Let E^∞ denote the infinite path space of E. Two paths $\alpha, \beta \in E^\infty$ are shift equivalent with lag $n \in \mathbb{Z}$, denoted $\alpha \sim_n \beta$, if there exists $N \in \mathbb{N}$ such that $\alpha_i = \beta_{i+n}$ for all $i \geq N$. Let

$$G = \{(\alpha, n, \beta) \in E^\infty \times \mathbb{Z} \times E^\infty : \alpha \sim_n \beta\}.$$

[2]A directed graph is *row-finite* if each vertex emits at most finitely many edges.

Next, define
$$G^{(2)} = \{((\alpha,n,\beta),(\gamma,m,\delta)) \in G \times G : \beta = \gamma\}$$
and let
$$(\alpha,n,\beta)(\beta,m,\delta) := (\alpha,n+m,\delta), \qquad (\alpha,n,\beta)^{-1} := (\beta,-n,\alpha).$$

Then G is a groupoid. The unit space $G^{(0)} = \{(\alpha,0,\alpha) \in G : \alpha \in E^\infty\}$ can be naturally identified with E^∞ and the range and source maps are given by $r(\alpha,n,\beta) = \alpha$ and $s(\alpha,n,\beta) = \beta$. It is shown in [KPRR97, Proposition 2.6] that G carries a topology making it into a locally compact Hausdorff r-discrete groupoid, called the "graph groupoid" associated to E.

Remark 1.21. The reason that the groupoids in Examples 1.19 and 1.20 look so similar is that they are both associated to generalizations of Cuntz-Krieger algebras [Dea96, KPRR97].

Haar measure is essential to the study of locally compact groups because it allows one to integrate. We will also want to integrate on groupoids and to do that we will need the following generalization of Haar measure.

Definition 1.22. A (left) *Haar system* on a locally compact Hausdorff groupoid G is a family $\lambda = \{\lambda^u\}_{u\in G^{(0)}}$ of non-negative Radon measures on G such that

(a) $\operatorname{supp}(\lambda^u) = G^u$ for all $u \in G^{(0)}$,[3]

(b) for $f \in C_c(G)$ the function
$$u \mapsto \int_G f(\gamma)d\lambda^u(\gamma)$$
on $G^{(0)}$ is in $C_c(G^{(0)})$; and

(c) for $\gamma \in G$ we have $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$. In other words, given $f \in C_c(G)$,
$$\int_G f(\gamma\eta)d\lambda^{s(\gamma)}(\eta) = \int_G f(\eta)d\lambda^{r(\gamma)}(\eta).$$

Given $u \in G^{(0)}$ we will use λ_u to denote $(\lambda^u)^{-1}$. In other words, given $f \in C_c(G)$,
$$\int_G f(\gamma)\lambda_u(\gamma) = \int_G f(\gamma^{-1})\lambda^u(\gamma).$$

[3]For a Borel measure μ on a topological space X the support of μ, denoted $\operatorname{supp}\mu$, is defined to be the largest (closed) subset of X for which every open neighborhood of every point of the set has positive measure.

The following is a technical lemma which we will use ever so often.

Lemma 1.23. *Given a groupoid G with Haar system $\{\lambda^u\}$ and a compact set $K \subset G$ then the set $\{\lambda^u(K)\}$ is bounded.*

Proof. Choose a compact neighborhood L of K and a positive function f which is one on K and zero off L. Then $\lambda^u(K) \leq \int_G f\lambda^u$ for all u and the function $u \mapsto \int_G f\lambda^u$ is continuous and compactly supported. The result follows. $\square$

Unlike Haar measure, Haar systems are not always guaranteed to exist and may not be unique in any reasonable sense. What's more, only groupoids with open range and source maps can have Haar systems. The following is asserted in [Ren80] and proved in [Sed86].

Proposition 1.24. *If G is a locally compact Hausdorff groupoid with a Haar system then the range and source maps are open.*

This is a good opportunity to mention a characterization of surjective open maps which we will use constantly and is stated and proved in [Wil07, Proposition 1.15].

Proposition 1.25. *Let $p : X \to Y$ be a continuous surjection between two topological spaces. Then p is an open map if and only if given a net $\{y_i\}_{i\in I}$ converging to $p(x)$ in Y, there is a subnet $\{y_{i_j}\}_{j\in J}$ and a net $\{x_j\}_{j\in J}$ indexed by the same set which converges to x in X, and which also satisfies $p(x_j) = y_{i_j}$.*

Because Haar systems will be necessary to build groupoid C^*-algebras we will usually assume that they exist. Luckily, for most of the groupoids that we are interested in there is a reasonable Haar system.

Example 1.26. Suppose X is a locally compact Hausdorff space and $G = X \times X$ is the associated trivial groupoid. Let λ be any measure on X with full support and define $\lambda^x = \delta_x \times \lambda$. Then it is straightforward to show that $\{\lambda^x\}$ is a Haar system for G. Now suppose we view X as the cotrivial groupoid. Then the collection of Dirac delta measures $\{\delta_x\}$ forms a Haar system for X. Since integration against δ_x is just evaluation it's easy to see that the continuity condition is satisfied and all of the other conditions follow from the fact that the operations are "cotrivial."

Example 1.27. Suppose H is a locally compact Hausdorff group acting on a locally compact Hausdorff space X and $G = H \times X$ is the associated transformation groupoid. Let λ be a Haar measure for H and define $\lambda^x = \lambda \times \delta_x$. Then $\{\lambda^x\}$ is a Haar system for G. We will always give transformation group groupoids this Haar system.

Example 1.28. The groupoids in both Example 1.19 and Example 1.20 can be given a Haar system by letting λ^u be counting measure on G^u for all $u \in G^{(0)}$ [Dea95, KPRR97].

The situation from Example 1.28 is actually much more generic. The following proposition is proved in [Ren80, Propositions 2.7,2.8].

Proposition 1.29. *Suppose G is an r-discrete groupoid.*

(a) *For any $u \in G^{(0)}$, G^u and G_u are discrete spaces.*

(b) *If $\{\lambda^u\}$ is a Haar system on G then each λ^u is a multiple of the counting measure.*

(c) *The following are equivalent:*

- **(i)** *G admits a Haar system,*
- **(ii)** *r and s are local homeomorphisms,*
- **(iii)** *the product map $G^{(2)} \to G$ is a local homeomorphism.*

1.1.1 The Stabilizer Subgroupoid

One slightly surprising fact is that a groupoid (potentially) contains many different groups.

Proposition 1.30. *Suppose G is a locally compact Hausdorff groupoid and $u \in G^{(0)}$. Then $S_u = G_u \cap G^u = \{\gamma \in G : r(\gamma) = s(\gamma) = u\}$, with the operations inherited from G, is a locally compact Hausdorff group which is closed in G.*

Proof. First, it's clear that $u \in S_u$ so that S_u is not empty. Now, every element in S_u has range and source u so that any two elements are composable. Thus the groupoid operation is everywhere defined on $S_u \times S_u$ and its associative because of the associativity condition in Definition 1.1. Given $\gamma \in S_u$, since $s(\gamma) = r(\gamma) = u$, we know from Proposition 1.5 that $\gamma u = u\gamma = \gamma$. Finally, given $\gamma \in S_u$ we have $\gamma^{-1}\gamma = s(\gamma) = u$ and $\gamma\gamma^{-1} = r(\gamma) = u$. Thus, S_u is a group.

Next suppose we have a net $\gamma_i \to \gamma$ in G such that $\gamma_i \in S_u$ for all i. The fact that the range and source maps are continuous implies $r(\gamma_i) \to r(\gamma)$ and $s(\gamma_i) \to s(\gamma)$. However, $r(\gamma_i) = s(\gamma_i) = u$ for all i so clearly, because G is Hausdorff, $r(\gamma) = s(\gamma) = u$. Thus S_u is closed and it follows that the relative topology on S_u is locally compact Hausdorff [Wil07, Lemma 1.26]. Finally, since the operations are continuous on G they are continuous on S_u. Thus S_u is a locally compact Hausdorff group. □

These groups will play an important role and are given their own special name.

Definition 1.31. Suppose G is a groupoid and $u \in G^{(0)}$ then the group

$$S_u = \{\gamma \in G : s(\gamma) = r(\gamma) = u\}$$

is known as the *stabilizer subgroup* of G at u. The set

$$S := \{\gamma \in G : s(\gamma) = r(\gamma)\} = \bigcup_{u \in G^{(0)}} S_u$$

is called the *stabilizer subgroupoid* of G. Well use p to denote the restriction of the range (and source) map to S. Oftentimes the word *isotropy* is used interchangeably with *stabilizer*.

Remark 1.32. Since S_u is a group we will generally denote elements of S_u and S by lowercase Roman letters, instead of the Greek letters used to denote generic elements of G.

Proposition 1.33. *Suppose G is a locally compact Hausdorff groupoid and let S be the stabilizer subgroupoid of G. Then S is a locally compact Hausdorff subgroupoid which is closed in G. Furthermore, S can be viewed as a bundle over $G^{(0)}$ with bundle map p whose fibres are the isotropy subgroups.*

Proof. Suppose $g, h \in S$. If g and h are composable then there exists $u \in G^{(0)}$ such that $g, h \in S_u$ and it follows that $gh \in S_u \subset S$. Similarly, we find that S is closed under the inverse operation. Since S is closed under the operations inherited from G it's clear that S is a groupoid in its own right, where $S^{(2)} = \{(g, h) \in G^{(2)} : g, h \in S\}$. Furthermore, since s and r are continuous, if $\gamma_i \to \gamma$ in G and $s(\gamma_i) = r(\gamma_i)$ for all i then $s(\gamma) = r(\gamma)$. Thus S is closed in G, and it follows that the relative topology makes S into a locally compact Hausdorff groupoid. The remaining statements of the proposition are clear. □

The stabilizer subgroupoid is a very important object. It can oftentimes tell us a lot about its parent groupoid. An even better situation is when the stabilizer subgroupoid is everything.

Definition 1.34. A *groupoid group bundle*, or *group bundle* for short, is a locally compact Hausdorff groupoid S such that the range and source maps are equal. We will denote the range (and source) map by p. We view S as a bundle over $S^{(0)}$ with bundle map p and denote the fibres by $S_u = p^{-1}(u)$ for all $u \in S^{(0)}$. We say that S is an *abelian group bundle* if S_u is an abelian group for all $u \in S^{(0)}$.

Remark 1.35. Groupoid group bundles are different than the kinds of group bundles one usually encounters. For instance, groupoid group bundles carry no kind of local triviality condition and the fibres can and will vary over the base space. Furthermore, the injection of the unit space into S always gives a continuous section of the bundle map.

Example 1.36. Suppose we have a locally compact Hausdorff group H and a topological space X. Then we can view $S = X \times H$ as a group bundle where the bundle map is just the projection onto the first factor. In this case the unit space can be identified with X and the groupoid operations are obvious.

It turns out that the existence of a Haar system on a group bundle S is equivalent to requiring the fibres of S vary "continuously." In order to make this notion precise we recall the following from [Wil07, Section H.1].

Definition 1.37. Let X be an arbitrary topological space and $\mathcal{C}(X)$ the collection of all closed subsets of X (including the empty set). Given a finite collection $\mathcal{F}$ of open sets of X and a compact subset K of X we define

$$\mathcal{U}(K;\mathcal{F}) := \{F \in \mathcal{C}(X) : F \cap K = \emptyset \text{ and } F \cap U \neq \emptyset \text{ for all } U \in \mathcal{F}\}.$$

The collection $\{\mathcal{U}(K;\mathcal{F})\}$ forms a basis for a compact Hausdorff topology on $\mathcal{C}(X)$ called the *Fell Topology*.

Proposition 1.38. *Suppose X is a locally compact space and let $\{F_i\}_i \in I$ be a net in $\mathcal{C}(X)$. Then $F_i \to F$ in $\mathcal{C}(X)$ if and only if*

(a) *given $t_i \in F_i$ such that $t_i \to t$, then $t \in F$, and*

(b) *if $t \in F$, then there is a subnet $\{F_{i_j}\}$ and $t_{i_j} \in F_{i_j}$ such that $t_{i_j} \to t$.*

Using Definition 1.37 to pin down the appropriate notion of continuity we can make the following

Definition 1.39. Suppose S is a locally compact Hausdorff group bundle. We say that S is *continuously varying* if given a net $\{u_i\}$ in $S^{(0)}$ such that $u_i \to u$ in $S^{(0)}$ then $S_{u_i} \to S_u$ with respect to the Fell topology. If G is a locally compact Hausdorff groupoid we say that G has *continuously varying stabilizers* if its stabilizer subgroupoid S is continuously varying.

At this point we can give some reasonable conditions for the existence of a Haar system for a group bundle. The following also provides a partial converse to Proposition 1.24.

Proposition 1.40. *Suppose S is a locally compact Hausdorff groupoid group bundle with bundle map p. The following are equivalent:*

(a) *S has a Haar system,*

(b) *p is open,*

(c) *S is continuously varying.*

Proof. It is shown in [Ren91, Lemma 1.3] that (a) and (b) are equivalent. Now suppose p is open and $u_i \to u$ in $S^{(0)}$. We will show $S_{u_i} \to S_u$ using Proposition 1.38. If $s_i \in S_{u_i}$ and $s_i \to s$ then $p(s_i) = u_i \to p(s)$. It follows that $p(s) = u$ and $s \in S_u$. Now suppose $s \in S_u$. Because p is open we can use Proposition 1.25 to pass to a subnet and find $s_{i_j} \in S_{u_{i_j}}$ such that $s_{i_j} \to s$. Thus $S_{u_i} \to S_u$.

Next suppose S varies continuously. We will show p is open using the characterization in Proposition 1.25. Let $u_i \to u$ be a net which converges in $S^{(0)}$ and suppose $s \in S_u$. Using Proposition 1.38 we can pass to a subnet and find $s_{i_j} \in S_{u_{i_j}}$ such that $s_{i_j} \to s$. Since $p(s_{i_j}) = u_{i_j}$ this proves p is open. □

Remark 1.41. We will pass to subnets frequently, and may even make use of sub-subnets. In order to avoid notational clutter we will usually relabel so that only one index is shown.

Remark 1.42. Given a locally compact groupoid group bundle S with a Haar system $\{\lambda^u\}$ each λ^u is a measure supported on the group $S_u = p^{-1}(u)$. It follows from the left invariance condition of Definition 1.22 that λ^u is a Haar measure on S_u. Thus a Haar system on S is just a "continuously varying" collection of Haar measures. It is straightforward to see that given another Haar system $\{\mu^u\}$ on S the unicity of Haar measure guarantees the existence of a continuous function $f : G^{(0)} \to \mathbb{C}$ such that $\lambda^u = f(u)\mu^u$ for all $u \in G^{(0)}$.

Example 1.43. Let T be the cone of length 1 and maximum radius 1 aligned on the positive x-axis as in Figure 1.2 and let p_T be the projection onto the x-axis. It is fairly clear that T is a continuously varying group bundle with each fibre equal to the circle group, after a scaling. Similarly let Z be the collection of line segments given by $y = nx$ for all $n \in \mathbb{Z}$ and $x \in [0, 1]$ as in Figure 1.2 and let p_Z be the projection onto the x-axis. Then Z is also a continuously varying group bundle and each fibre is equal to the integers, again after a scaling.

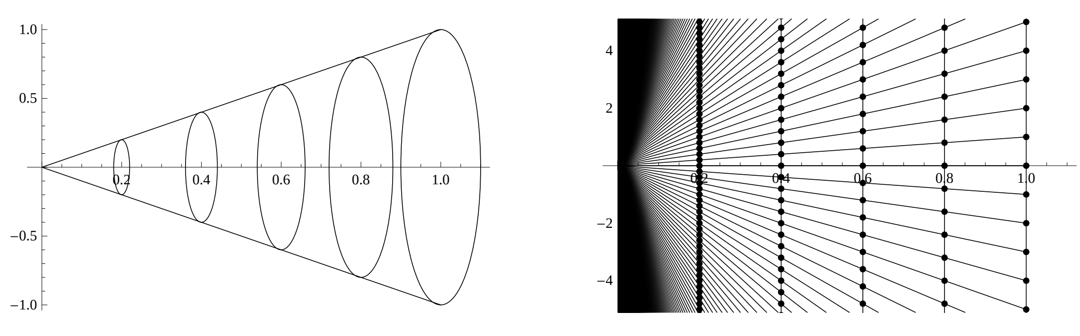

Figure 1.2: Examples of continuously varying group bundles.

Remark 1.44. If G is a locally compact Hausdorff groupoid then Proposition 1.40 implies that G has continuously varying stabilizers if and only if the stabilizer subgroupoid has a Haar system. This is a natural hypothesis and will be frequently invoked. However, it's important to understand that this is a very strong condition. For instance, if $G = H \times X$ is a transformation groupoid associated to an action of H on X then it's easy to see that the stabilizer subgroup S_x is isomorphic to $\{s \in H : s \cdot x = x\}$. In other words, the stabilizer subgroup S_x is exactly the stabilizer subgroup of H at x. It's also straightforward to show that the stabilizers vary continuously in G if and only if they vary continuously in H. However, most group actions do not have continuously varying stabilizers; even really nice actions. For example, if $\mathbb{T}$ acts on $\mathbb{R}^2$ by rotation then the stabilizers are discontinuous at the origin.

1.1.2 The Orbit Groupoid

Yet another important class of groupoid are those that are "opposite" of Definition 1.34. In other words, groupoids whose isotropy subgroupoid is as trivial as possible.

Definition 1.45. A groupoid G is called *principal* if $r(\gamma) = s(\gamma)$ implies $\gamma \in G^{(0)}$ for all $\gamma \in G$.

Any groupoid gives rise to a canonical principal groupoid.

Definition 1.46. Suppose G is a groupoid. We define the *orbit equivalence relation* on $G^{(0)}$ to be given by $u \sim v$ if and only if there exists $\gamma \in G$ such that $u = r(\gamma)$ and $v = s(\gamma)$. We define the *orbit groupoid* to be $R = \{(u, v) \in G \times G : u \sim v\}$ where $R^{(2)} = \{((u, v), (w, z)) \in R \times R : v = w\}$ and the operations are given by

$$(u, v)(v, w) := (u, w), \qquad\qquad (u, v)^{-1} := (v, u).$$

Finally, we call the map $\pi = (r, s) : G \to R$ defined by $\pi(\gamma) = (r(\gamma), s(\gamma))$ the *canonical homomorphism*.

Proposition 1.47. *Suppose G is a groupoid. Let $\sim$ be the orbit equivalence relation, R the orbit groupoid, and $\pi = (r, s)$ the canonical homomorphism from Definition 1.46. Then $\sim$ is an equivalence relation, and R is a principal groupoid. The unit space of R can be identified with $G^{(0)}$ and under this identification $r(u, v) = u$ and $s(u, v) = v$. Furthermore, π is a surjective groupoid homomorphism.*

Proof. Since $u = r(u) = s(u)$ we see that $\sim$ is reflexive. We know $\sim$ is symmetric because if $u = r(\gamma)$ and $v = s(\gamma)$ then $u = s(\gamma^{-1})$ and $v = r(\gamma^{-1})$. Finally if $u = r(\gamma)$, $v = s(\gamma) = r(\eta)$ and $w = s(\eta)$ then, citing Proposition 1.5, $u = r(\gamma\eta)$ and $v = s(\gamma\eta)$. Thus $\sim$ is transitive.

As for the groupoid R, this is exactly the situation in Example 1.15. Observe that the composition operation is well defined, i.e. it maps into R, because of the transitivity of $\sim$. Similarly the inverse is well defined because of symmetry. Now, suppose

$$((x,y),(y,w)),((y,w),(w,z)) \in R^{(2)},$$

then it's clear that

$$((x,w),(w,z)),((x,y),(y,z)) \in R^{(2)},$$

and that

$$((x,y)(y,w))(w,z) = (x,z) = (x,y)((y,w)(w,z)).$$

Therefore associativity is satisfied, and it's obvious that involution is also satisfied. Finally, suppose $(x,y) \in R$. Clearly $((x,y),(y,x)) \in R^{(2))}$ and given $(y,w) \in R$

$$\begin{aligned}(y,x)((x,y)(y,w)) &= (y,x)(x,w) = (y,w)\\ ((x,y)(y,w))(w,y) &= (x,w)(w,y) = (x,y).\end{aligned}$$

Thus cancellation also holds and R is a groupoid. Given $(x,y) \in R$ we have $r((x,y)) = (x,y)(y,x) = (x,x)$. From here it is clear that $G^{(0)} = r(G) = \{(x,x) : x \in G^{(0)}\}$. This set is trivially identifiable with $G^{(0)}$ and under this identification $r(x,y) = x$. Similarly $s(x,y) = y$. Lastly, if $r(x,y) = s(x,y)$ then $x = y$ and $(x,y) \in R^{(0)}$, making R principal.

Observe that $(r(\gamma), s(\gamma)) \in R$ for all $\gamma \in G$ so π is well defined. Furthermore if $(\gamma,\eta) \in G^{(2)}$ then $s(\gamma) = r(\eta)$ so that $((r(\gamma),s(\gamma)),(r(\eta),s(\eta))) \in R^{(2)}$ and

$$\begin{aligned}\phi(\gamma\eta) &= (r(\gamma\eta), s(\gamma\eta)) = (r(\gamma), s(\eta))\\ &= (r(\gamma),s(\gamma))(r(\eta),s(\eta)) = \phi(\gamma)\phi(\eta).\end{aligned}$$

Finally, it is clear from the definition of R that π is surjective. □

Proposition 1.48. *If G is a principal groupoid then the canonical homomorphism $\pi : G \to R$ is an isomorphism.*

Proof. We already know that π is a surjective homomorphism. Suppose $\pi(\gamma) = \pi(\eta)$ for $\gamma, \eta \in G$. Then $r(\gamma) = r(\eta)$ and $s(\gamma) = s(\eta)$. This means that γ and η^{-1} are composable and that

$$s(\gamma\eta^{-1}) = r(\eta) = r(\gamma) = r(\gamma\eta^{-1}).$$

Therefore $\gamma\eta^{-1} \in G^{(0)}$. If we compose $\gamma\eta^{-1}$ on both sides of the equation $\eta = \eta$ we get

$$\gamma = \gamma\eta^{-1}\eta = \eta$$

where the left hand equality comes from cancellation and the right hand equality

comes from the fact that $\gamma\eta^{-1}$ is a unit. Thus π is injective and therefore an isomorphism. □

Remark 1.49. Proposition 1.48 states that every principal groupoid is isomorphic to its orbit groupoid. However, the orbit groupoid is determined by the orbit equivalence relation. Thus, disregarding topology for a moment, every principal groupoid is (isomorphic to) one of the groupoids defined in Example 1.15.

The isotropy groupoid of a locally compact Hausdorff groupoid is naturally a locally compact Hausdorff groupoid. Unfortunately the situation is not so simple for the orbit groupoid.

Definition 1.50. Suppose G is a locally compact Hausdorff groupoid and R is the orbit groupoid determined by G. We denote R with the relative topology of $G^{(0)} \times G^{(0)}$ by R_P. We denote R with the quotient topology with respect to the canonical homomorphism $\pi = (r, s)$ by R_Q. When R is used as a topological groupoid it will always mean R_Q.

Proposition 1.51. *Suppose G is a locally compact Hausdorff groupoid. Then the topology on R_Q is finer than the topology on R_P. Furthermore R_P and R_Q are both Hausdorff and the map π is continuous as a function into both R_Q and R_P. Next, R_P is a topological groupoid and, if either G is second countable or R_Q is locally compact, then R_Q is a topological groupoid. Finally, if G has open range and source then the range and source maps are open as maps on R_P and R_Q.*

Proof. First we will show $\pi : G \to R_P$ is continuous. Suppose $\gamma_i \to \gamma \in G$. Since the range and source maps are continuous we have $(r(\gamma_i), s(\gamma_i)) \to (r(\gamma), s(\gamma))$ in $G^{(0)} \times G^{(0)}$ and hence in R_P. Since R_Q has the quotient topology determined by π, clearly $\pi : G \to R_Q$ must be continuous.

Next, suppose O is open in R_P, then $\pi^{-1}(O)$ is open in G, but this implies that O is open in R_Q. Thus R_Q has a finer topology than R_P. Furthermore, since any subset of a Hausdorff space inherits a Hausdorff topology, R_P is Hausdorff. This implies R_Q is Hausdorff as well, since it carries a finer topology.

It's pretty easy to see that the operations on R_P are continuous with respect to the product topology. After all $(u_i, v_i) \to (u, v)$ in R_P if and only if $u_i \to u$ and $u_i \to u$. Proving that the operations are continuous on R_Q takes more work, and more hypotheses. Let $I : G \to G$ be defined by $I(\gamma) = \gamma^{-1}$ and consider $\pi \circ I : G \to R_Q$. It's easy to see that I is a continuous map and that if $\pi(\gamma) = \pi(\eta)$ then $\pi \circ I(\gamma) = \pi \circ I(\eta)$. Thus $I \circ \pi$ factors to a continuous map from R_Q into R_Q and clearly this factorization is nothing more than the inversion operation on R_Q. We would like to use the same argument with the multiplication. The main issue is the following

Claim. The map $\pi \times \pi : G \times G \to R_Q \times R_Q$ is a quotient map.

Proof of Claim. It is known [Mic68, Section 8] that the product of a quotient map with itself need not be quotient. However, there are rather minimal conditions on R_Q which will guarantee that $\pi \times \pi$ is a quotient map. We know from [Mic68, Theorem 1.5] that if G and $R_Q \times R_Q$ are Hausdorff k-spaces[4] then $\pi \times \pi$ will be a quotient map. It's clear that G and $R_Q \times R_Q$ are Hausdorff and, since locally compact spaces are always k-spaces, all that is left is to show that $R_Q \times R_Q$ is a k-space. If, on one hand, R_Q is locally compact then $R_Q \times R_Q$ is locally compact and we are done. On the other hand, suppose G is second countable. Then by choosing a countable basis of compact neighborhoods we can find a countable collection $\{K_n\}$ of compact sets which cover G and have the property that a set $A \subset G$ is closed if and only if $A \cap K_n$ is closed for all n. Such a space is known as a k_ω-space. It follows from [Mic68, Remark 7.5] that quotients and products of k_ω-spaces are k_ω-spaces so that we can conclude $R_Q \times R_Q$ is a k_ω-space. Thus assuming either G is second countable or R_Q is locally compact proves our claim. □

Suppose the claim is satisfied. It's straightforward to see that $R_Q^{(2)}$ is closed in $R_Q \times R_Q$ and, since $G^{(2)}$ is just the set of those elements whose range and sources match up, that $G^{(2)} = (\pi \times \pi)^{-1}(R^{(2)})$. It is also straightforward to see that the restriction of a quotient map to the inverse image of a closed set results in a quotient map. Thus the restriction $\pi \times \pi : G^{(2)} \to R_Q^{(2)}$ is a quotient map. Let $M : G^{(2)} \to G$ be given by $M(\gamma, \eta) = \gamma\eta$ and consider $\pi \circ M : G^{(2)} \to R_Q$. Then it's easy to see that if $\pi \times \pi(\gamma, \eta) = \pi \times \pi(\gamma', \eta')$ then $\pi \circ M(\gamma, \eta) = \pi \circ M(\gamma', \eta')$ so that $\pi \circ M$ factors to a continuous map from $R_Q^{(2)}$ into R_Q. Furthermore, this map is clearly the composition operation on R_Q implying that composition is continuous.

Finally, suppose G has open range and source maps, that $u_i \to u$ in $G^{(0)}$ and that $(u, v) \in R$. First, choose any $\gamma \in G$ such that $\pi(\gamma) = (u, v)$. Since the range on G is open we can pass to a subnet and find $\gamma_i \to \gamma$ such that $r(\gamma_i) = u_i$. Then $\pi(\gamma_i) \to \pi(\gamma) = (u, v)$ in both R_P and R_Q and $r(\pi(\gamma_i)) = u_i$. Thus r is open on R_P and R_Q. The proof that s is open is similar. □

Remark 1.52. The astute reader will have noticed that there are several things missing from Proposition 1.51. For instance, neither R_P nor R_Q are necessarily locally compact. What's more, in extreme cases it is not clear if R_Q is even a topological groupoid. On the other hand, the operations on R_Q are always continuous if G is second countable, which includes almost all of the examples that we will care about. It's also nice to note that if the topology on R_Q is well behaved (i.e. locally compact) then the operations are continuous then as well. However, even in this case R_Q may not have a Haar system when G does.

[4] A topological space X is called a *k-space* if a set $A \subset X$ is closed whenever $A \cap K$ is closed in K for every compact $K \subset X$. Such a space is also called *compactly generated*.

Interestingly enough, given a groupoid G there is a duality between its orbit groupoid and its stabilizer subgroupoid. The following is stated in [Ren91, Remark 1.2].

Proposition 1.53. *Suppose G is a locally compact Hausdorff groupoid. Then G has continuously varying stabilizers if and only if the canonical map $\pi = (r, s)$ is open onto R_Q. Furthermore, under these conditions R_Q is a locally compact Hausdorff groupoid, and if G is second countable then R_Q is also.*

Proof. Let G be a locally compact Hausdorff groupoid, S the stabilizer subgroupoid, and p the bundle map for S. First, suppose G has continuously varying stabilizers so that Proposition 1.40 implies that p is open. Let O be an open set in G. We want to show that $\pi^{-1}(\pi(O))$ is open. Well

$$\pi^{-1}(\pi(O)) = \{\gamma \in G : \exists\, \eta \in O \text{ s.t. } \pi(\gamma) = \pi(\eta)\}.$$

However, if $\pi(\eta) = \pi(\gamma)$ then $\gamma\eta^{-1} \in S$. Therefore, if $\gamma \in \pi^{-1}(\pi(O))$ then $\gamma \in S \cdot O = \{s\eta : s \in S, \eta \in O, p(s) = r(\eta)\}$. What's more, if $s\eta \in S \cdot O$ then $\pi(s\eta) = \pi(\eta)$ so that $s\eta \in \pi^{-1}(\pi(O))$. It follows that $\pi^{-1}(\pi(O)) = S \cdot O$. Now suppose $S \cdot O$ is not open so that there exists $s \in S, \gamma \in O$ and a net $\{\gamma_i\}$ such that $\gamma_i \to s\gamma$ and for all i we have $\gamma_i \notin S \cdot O$. Now, $r(\gamma_i) \to r(\gamma)$ and $p(s) = r(\gamma)$ so that, using Proposition 1.25, we can pass to a subnet, relabel, and find $s_i \in S_{r(\gamma_i)}$ such that $s_i \to s$. It follows that $s_i^{-1}\gamma_i \to s^{-1}s\gamma = \gamma$. Since O is open this implies $s_i^{-1}\gamma_i \in O$ eventually and this is a contradiction since $\gamma_i = s_i(s_i^{-1}\gamma_i)$. Hence $S \cdot O$ is open so that π is open onto R_Q.

Next, let $\pi : G \to R_Q$ be open. Suppose $u_i \to u$ is a convergent net in $G^{(0)}$ and that $p(s) = u$. Observe that $\pi(u_i) = (u_i, u_i) \to \pi(u) = (u, u)$ and that $\pi(s) = (u, u)$. Since π is open we can use Proposition 1.25 to pass to a subnet, relabel, and find $\gamma_i \in G$ such that $\gamma_i \to s$ and $\pi(\gamma_i) = (u_i, u_i)$. However, this implies $\gamma_i \in S_{u_i}$ for all i. Thus, using Proposition 1.25 again, p is an open map.

Suppose π is open onto R_Q, or equivalently that the stabilizers vary continuously. It follows that R_Q is locally compact, since the image of a basis of compact neighborhoods under π will be a basis of compact neighborhoods for R_Q. In this case, Proposition 1.51 implies that the operations on R_Q are continuous. Furthermore, since the image of a countable basis under π will be a countable basis, if G is second countable then so is R_Q. $\square$

Remark 1.54. It is not necessary for the stabilizers to vary continuously for R_Q to be a locally compact Hausdorff (topological) groupoid. For instance, consider $\mathbb{T}$ acting on the closed unit ball in $\mathbb{R}^2$ by rotation. Then R_Q is compact since it's the continuous image of a compact space, and is therefore a topological groupoid. However, the stabilizers are clearly discontinuous at the origin.

1.2 Groupoid Spaces

The notion of a groupoid action on a space is a straightforward generalization of group actions. The only caveat is that the action is only "partially defined" in the same sense that the groupoid multiplication is only partially defined. Once again, much of this section is inspired by [Muh].

Definition 1.55. Suppose G is a groupoid and X is a set. We say that G acts (on the left) of X, and that X is a *left G-space*, if there is a surjection $r_X : X \to G^{(0)}$ and a map $(\gamma, x) \mapsto \gamma \cdot x$ from $G * X := \{(\gamma, x) \in G \times X : s(\gamma) = r_X(x)\}$ to X such that

(a) if $(\eta, x) \in G * X$ and $(\gamma, \eta) \in G^{(2)}$, then $(\gamma\eta, x), (\gamma, \eta \cdot x) \in G * X$ and

$$\gamma \cdot (\eta \cdot x) = \gamma\eta \cdot x,$$

(b) and $r_X(x) \cdot x = x$ for all $x \in X$.

Right actions and right G-spaces are defined similarly except we use s_X to denote the map from X to $G^{(0)}$ and we define the action on the set $X * G := \{(x, \gamma) \in X \times G : s_X(x) = r(\gamma)\}$.

Remark 1.56. Given a left G-space we call $r_X(x)$ the range of x and in a right G-space $s_X(x)$ is the source of x. In order to avoid notational clutter we will almost always drop the subscripts on r_X and s_X. When an action is not specified to act on the right or the left we will assume that it acts on the left. (Unless it acts on the right, of course.)

Definition 1.57. Let G be a groupoid acting on both X and Y. A map $\phi : X \to Y$ is *G-equivariant* if and only if $r_X(x) = r_Y(\phi(x))$ and $\phi(\gamma \cdot x) = \gamma \cdot \phi(x)$ for all $x \in X$ and $\gamma \in G_{r(x)}$.

Remark 1.58. Suppose G is a groupoid, X is a G-space, $H \subset G$ and $A \subset X$. We will use the notation

$$H \cdot A := \{\gamma \cdot x : \gamma \in H, x \in A, s(\gamma) = r(x)\}.$$

As in Remark 1.3, it's important to realize that $H \cdot A$ may be poorly behaved. For instance if $s(H) \cap r(A) = \emptyset$ then $H \cdot A$ is empty.

Proposition 1.59. *Suppose G is a groupoid and X is a left G-space.*

(a) *Given $\gamma \in G$ and $x \in X$ such that $s(\gamma) = r(x)$ we have $r(\gamma \cdot x) = r(\gamma)$.*

(b) *Given $\gamma \in G$ and $x \in X$ such that $s(\gamma) = r(x)$ we have $\gamma^{-1} \cdot (\gamma \cdot x) = x$.*

(c) *Given $\gamma \in G$ and $x, y \in X$ such that $s(\gamma) = r(x)$ and $y = \gamma \cdot x$ we have $\gamma^{-1} \cdot y = x$.*

Similar statements hold if X is a right G-space.

Proof. Part **(a)**: If $s(\gamma) = r(x)$ then $(\gamma, x) \in G * X$. Furthermore, $(r(\gamma), \gamma) \in G^{(2)}$ so that by Definition 1.55 we have $(r(\gamma), \gamma \cdot x) \in G * X$. However, this implies

$$s(r(\gamma)) = r(\gamma) = r(\gamma \cdot x).$$

Part **(b)**: Given $(\gamma, x) \in G * X$ we have

$$\begin{aligned} \gamma^{-1} \cdot (\gamma \cdot x) &= (\gamma^{-1}\gamma) \cdot x = s(\gamma) \cdot x \\ &= r(x) \cdot x = x. \end{aligned}$$

Part **(c)**: Given $(\gamma, x) \in G * X$ and $y = \gamma \cdot x$ observe that $r(y) = r(\gamma \cdot x) = r(\gamma)$. Therefore we can act on both sides by γ^{-1} and use part (b) to obtain

$$\gamma^{-1} \cdot y = \gamma^{-1} \cdot (\gamma \cdot x) = x.$$

The corresponding statements for a right G action are proved similarly. □

The appropriate topological notion of a G-space is almost exactly what one would think.

Definition 1.60. Suppose G is a topological groupoid and X is a left G-space equipped with a topology. We say that G acts continuously on the left of X, and call X a *continuous left G-space*, if the maps $r_X : X \to G^{(0)}$ and $(\gamma, x) \mapsto \gamma \cdot x$ from $G * X$ to X are continuous. Continuous right G-spaces are defined similarly. We will call X a *strongly continuous left G-space* (resp. *strongly continuous right G-space*) if r_X (resp. s_X) is an open map. Furthermore, we will often refer to a strongly continuous G-space as a *strong G-space*.

Remark 1.61. Definition 1.60 is different from the usual definition found in the literature. It is standard to make the requirement that r_X (or s_X) be open part of the definition of a G-space and to forgo the notion of a "strongly continuous G-space" altogether. The author has chosen to introduce some new terminology and break from the literature for (at least) three reasons. The first is that the material in Section 2.1 does not require r_X to be open. The second is that if r_X is required to be open then technically groupoids without Haar systems might not act "continuously" on themselves. Lastly, and most importantly, given a groupoid dynamical system (A, G, α) as in Section 3.2, we will be interested in the induced action of G on $\operatorname{Prim} A$. However, unless A is a *continuous* C^*-bundle the range map on $\operatorname{Prim} A$ will not be open. Since

it will be necessary to deal with these actions at some point, we have decided to make this distinction part of the definition.

Example 1.62. Suppose the locally compact group H acts on a locally compact space X. Then, if we view H as a groupoid with only one unit and let the range map for X map onto this point, it's easy to see that H is a strongly continuous groupoid action on X.

Example 1.63. The following is a particularly important example of a groupoid action. Suppose G is a locally compact Hausdorff groupoid and H is a closed subgroupoid. Let $X = s^{-1}(H^{(0)})$ and give X the relative topology. Since H is closed in G, $H^{(0)}$ is closed in $G^{(0)}$ so that X is closed in G and must be locally compact Hausdorff. Let $s : X \to H^{(0)}$ be the restriction of the source map on G to X. Given $(\xi, \eta) \in X * H$ we have $s(\xi) = r(\eta)$ so that we can define $\xi \cdot \eta = \xi\eta$. Observe that $s(\xi\eta) = s(\eta) \in H^{(0)}$ so that the action is well defined on X. Since this action is defined via the groupoid operation it is straightforward to see that X is a continuous right G-space. Furthermore, if G has open range and source then, because X is a saturated[5] closed set of G with respect to s, it follows that the restriction of s to X is open and in this case X is a strongly continuous right G-space. We can also define an analogous left G-space in a similar fashion.

Example 1.64. This is a special case of Example 1.63 but deserves to be singled out. Suppose G is a locally compact Hausdorff groupoid. If we treat G as a closed subgroupoid of itself and use the construction from Example 1.63 we find that $X = G$ and that there is a continuous right action of G on itself defined by multiplication. Furthermore, if G has open range and source then clearly this action is strongly continuous.

Example 1.65. Suppose G is a locally compact Hausdorff groupoid. Let $r : G^{(0)} \to G^{(0)}$ be the identity map (or the restriction of r to $G^{(0)}$, whichever you prefer). Given $(\gamma, u) \in G * G^{(0)}$ define

$$\gamma \cdot u := r(\gamma) = \gamma u \gamma^{-1}.$$

It is straightforward to check that this defines a groupoid action and that with this action $G^{(0)}$ is a strong G-space. As usual, we can let G act on the right of $G^{(0)}$ in a similar fashion.

Example 1.66. Suppose G is a locally compact Hausdorff groupoid and S is the stabilizer subgroupoid. Let $p : S \to G^{(0)}$ be the restriction of the range map to S (or the bundle map associated to S, whichever you prefer). Given $(\gamma, s) \in G * S$ define

$$\gamma \cdot s := \gamma s \gamma^{-1}.$$

[5] Given a surjective map $f : X \to Y$ a set $A \subset X$ is saturated if $A = f^{-1}(f(A))$.

This action is well defined since $(\gamma, s) \in G * S$ implies $s(\gamma) = p(s)$. Furthermore, it's easy to see that $\gamma \cdot \eta \cdot s = \gamma\eta \cdot s$ and that $p(s) \cdot s = s$. It's fairly clear that the action is continuous so that S is a continuous G-space. Furthermore, if the stabilizers of G vary continuously then p is open and S is a strong G-space. As usual, we can let G act on the right of S in a similar fashion.

As with group actions, the action of a groupoid on a space defines an equivalence relation, and (fortunately) the quotient map associated to this equivalence relation is open whenever the range map on G is open. This is true even if X is not a strongly continuous G-space.

Definition 1.67. Suppose X is a (left) G-space. Define the *orbit equivalence relation* on X determined by G to be $x \sim y$ if and only if there exists $\gamma \in G$ such that $\gamma \cdot x = y$. The quotient space with respect to this relation is denoted X/G, the elements of X/G are denoted $G \cdot x$, and the canonical quotient map is (often) denoted by q. When X is a continuous G-space we will give X/G the quotient topology with respect to q. When X is a right G-space the orbit equivalence relation is defined similarly and we will use exactly the same notation.

In some cases X will be both a left G-space and a right H-space and in these situations we will denote the orbit space with respect to the G action by $G\backslash X$ and the orbit space with respect to the H action by X/H and we will denote elements of the orbit space by $G \cdot x$ and $x \cdot H$, respectively. It will occasionally be useful to denote the orbit $G \cdot x$ or $x \cdot G$ by $[x]$ to conserve notation.

Remark 1.68. Since the orbit equivalence relation on $G^{(0)}$ with respect to G as defined in Definition 1.46 is exactly the orbit equivalence relation on $G^{(0)}$ with respect to G when we view $G^{(0)}$ as a G-space via Example 1.65, there is no problem reusing the orbit equivalence relation terminology.

Proposition 1.69. *Let G be a locally compact Hausdorff groupoid and suppose X is a continuous G-space. Then the orbit equivalence relation defined in Definition 1.67 is an equivalence relation. If the range and source maps for G are open then the canonical quotient map $q : X \to X/G$ is open and X/G is locally compact. In particular, this is true if G has a Haar system. Furthermore, in this case if X is second countable then X/G is second countable.*

Proof. We will assume that G is a left G-space, but the proof is entirely analogous when G acts on the right. Since $r(x) \cdot x = x$ it is clear that $\sim$ is reflexive. Given $x, y \in X$ if $x \sim y$ then there exists $\gamma \in G$ such that $y = \gamma \cdot x$. However it follows from Proposition 1.59 that $\gamma^{-1} \cdot y = x$ and $y \sim x$. Finally if $x \sim y$ and $y \sim z$ then find $\gamma, \eta \in G$ such that $y = \gamma \cdot x$ and $z = \eta \cdot y$. Then $z = (\eta\gamma) \cdot x$ and $x \sim z$. Thus $\sim$ is an equivalence relation.

Now suppose G has open range and source maps. By Proposition 1.24 this is true whenever G has a Haar system. Since X/G has the quotient topology it suffices to see that $q^{-1}(q(O)) = G \cdot O$ is open whenever O is. Suppose x_i is a net in X such that $x_i \to \gamma \cdot x$ where $\gamma \in G$ and $x \in O$. Let $u_i = r(x_i)$, $u = r(\gamma \cdot x) = r(\gamma)$ and observe that $u_i \to u$. Since the range on G is an open map we can pass to a subnet, reindex, and find $\gamma_i \in G$ such that $\gamma_i \to \gamma$ and $r(\gamma_i) = u_i$ for all i. It follows then that $\gamma_i^{-1} \to \gamma^{-1}$ and, since the group action is continuous

$$\gamma_i^{-1} \cdot x_i \to \gamma^{-1} \cdot (\gamma \cdot x) = x.$$

However, O is open so eventually $\gamma_i^{-1} \cdot x_i \in O$. Thus, eventually, $x_i = \gamma_i \cdot (\gamma_i^{-1} \cdot x_i) \in G \cdot O$. This suffices to show that $G \cdot O$, and hence q, is open. Finally, the open image of a locally compact space is locally compact, since a basis of compact neighborhoods will map to a basis of compact neighborhoods. The same argument shows that if X is second countable then so is X/G. □

Just as in the group case we can associate a groupoid to a groupoid action.

Definition 1.70. Suppose G is a locally compact Hausdorff groupoid which acts continuously on the left of a locally compact Hausdorff space X. The *transformation groupoid* associated to G and X is the space $G \ltimes X = \{(\gamma, x) \in G \times X : r(\gamma) = r(x)\}$ with

$$G \ltimes X^{(2)} = \{((\gamma, x), (\eta, y)) \in (G \ltimes X) \times (G \ltimes X) : y = \gamma^{-1} \cdot x\}$$

and, when $((\gamma, x), (\eta, y)) \in G \rtimes X^{(2)}$, the operations

$$(\gamma, x)(\eta, y) = (\gamma\eta, x), \qquad\qquad (\gamma, x)^{-1} = (\gamma^{-1}, \gamma^{-1} \cdot x).$$

When G acts on the right of X the transformation groupoid is defined in an analogous fashion and is denoted $X \rtimes G$.

This groupoid is generally well behaved, as we will see after we prove the following utility lemma.

Lemma 1.71. *Suppose that X is a locally compact space and C is a closed subset of X. Given $f \in C_c(C)$ we can extend f to a function in $C_c(X)$.*

Proof. Now, if everything is second countable then X is normal so we can use the usual Tietze Extension Theorem. If we want to work with the nonseparable case we will need to make the following local argument. First, let U be some neighborhood of $K = \operatorname{supp} f$ in X with compact closure. Since C is closed in X it follows that $C \cap \overline{U}$ is a closed set in $\overline{U}$ and is therefore compact. We can now use the Tietze Extension Theorem for compact sets [Wil07, Lemma 1.42] to find a function $g \in C_c(X)$ such

that g is equal to f on $C\cap\overline{U}$. Observe that U is an open neighborhood of the compact set K so that we may use Urysohn's Lemma for compact sets [Wil07, Lemma 1.41] to find $h \in C_c(X)$ such that h is one on K and h is zero off U. Consider hg. Given $x \in K \subset C\cap\overline{U}$ we have $h(x) = 1$, and $g(x) = f(x)$. Thus $hg = f$ on K. If $x \in C \setminus K$ then $f(x) = 0$ and there are two cases to consider. In the first case $x \in U$ so that $x \in C \cap \overline{U}$ and $g(x) = f(x) = 0$. Otherwise $x \notin U$ so that $h(x) = 0$. In either case $hg(x) = 0$ so that hg is an extension of $f \in C_c(C)$ to $C_c(X)$. □

Transformation groupoids are very similar to their group action analogues. For instance, they have a Haar system as long as G does, even if the action of G is not strongly continuous.

Proposition 1.72. *Suppose G is a locally compact Hausdorff groupoid acting continuously on a locally compact Hausdorff space X. Then the transformation groupoid $G \ltimes X$ is a locally compact Hausdorff groupoid which is second countable if G and X are. The unit space $G \ltimes X^{(0)}$ can be naturally identified with X and under this identification*

$$r(\gamma, x) = x, \qquad\qquad s(\gamma, x) = \gamma^{-1} \cdot x.$$

The range and source maps are open if the range and source maps of G are open. Furthermore, if G has a Haar system $\{\lambda^u\}$ then $\mu^x = \lambda^{r(x)} \times \delta_x$ is a Haar system for $G \ltimes X$.

Proof. Suppose $((\gamma, x), (\eta, y)), ((\eta, y), (\xi, z)) \in G \ltimes X^{(2)}$. Then

$$(\gamma\eta)^{-1} \cdot x = \eta^{-1} \cdot (\gamma^{-1} \cdot x) = \eta^{-1} \cdot y = z$$

and we have assumed $\gamma^{-1} \cdot x = y$ so that $((\gamma\eta, x), (\xi, z)), ((\gamma, x), (\eta\xi, y)) \in G \ltimes X^{(2)}$. Furthermore we clearly have

$$(\gamma, x)((\eta, y)(\xi, z)) = (\gamma\eta\xi, x) = ((\gamma, x)(\eta, y))(\xi, z).$$

Next we calculate that

$$((\gamma, x)^{-1})^{-1} = (\gamma^{-1}, \gamma^{-1} \cdot x)^{-1} = (\gamma, \gamma \cdot (\gamma^{-1} \cdot x)) = (\gamma, x).$$

Finally suppose $(\gamma, x) \in G \ltimes X$. Then it's easy to see $((\gamma^{-1}, \gamma^{-1} \cdot x), (\gamma, x)) \in G \ltimes X^{(2)}$ and if $((\gamma, x), (\eta, y)) \in G \ltimes X^{(2)}$ then

$$\begin{aligned}(\gamma^{-1}, \gamma^{-1} \cdot x)((\gamma, x)(\eta, y)) &= (\gamma^{-1}, \gamma^{-1} \cdot x)(\gamma\eta, x)\\ &= (\gamma^{-1}\gamma\eta, \gamma^{-1} \cdot x)\\ &= (\eta, y).\end{aligned}$$

Similarly $((\gamma, x)(\eta, y))(\eta, y)^{-1} = (\gamma, x)$. Thus $G \ltimes X$ is a groupoid. We can now calculate

$$\begin{aligned}(\gamma^{-1}, \gamma^{-1} \cdot x)(\gamma, x) &= (s(\gamma), \gamma^{-1} \cdot x) = (r(\gamma^{-1} \cdot x), \gamma^{-1} \cdot x) \\ (\gamma, x)(\gamma^{-1}, \gamma^{-1} \cdot x) &= (r(\gamma), x) = (r(x), x).\end{aligned}$$

It follows that $G \ltimes X^{(0)} = r(G \ltimes X) = \{(r(x), x) : x \in X\}$ and the identification with X is obvious. Furthermore, under this identification, the previous calculations show that $r(\gamma, x) = x$ and $s(\gamma, x) = \gamma^{-1} \cdot x$.

Observe that, since the range maps for both G and X are continuous, the set $G \ltimes X$ is closed in $G \times X$ and is therefore locally compact Hausdorff. Clearly $G \ltimes X$ is second countable if G and X are. Furthermore, since the topology on $G \ltimes X$ is inherited from the product topology and because the operations on G and the action on X are continuous, it's easy to see that $G \ltimes X$ is a topological groupoid.

Now we show that the range and source maps of $G \ltimes X$ are open if the range and source maps of G are open. Suppose $x_i \to x$ and $(\gamma, x) \in G \ltimes X$. Since $r(\gamma) = r(x)$ and $r(x_i) \to r(x)$ we can pass to a subnet, reindex, and find $\gamma_i \in G$ such that $\gamma_i \to \gamma$ and $r(\gamma_i) = r(x)$. Since $(\gamma_i, x_i) \in G \ltimes X$ and $(\gamma_i, x_i) \to (\gamma, x)$ we see that r is open. Since s is the composition of r with the inversion map s must be open as well.

Next, suppose $\{\lambda^u\}$ is a Haar system for G and for all $x \in X$ define $\mu^x = \lambda^{r(x)} \times \delta_x$ where δ_x is the Dirac delta measure at x. Then clearly μ^x is a non-negative Radon measure and it's fairly easy to see that $\operatorname{supp} \mu^x = \operatorname{supp} \lambda^{r(x)} \times \operatorname{supp} \delta_x = G^{r(x)} \times \{x\}$. However, it's also clear that $G \ltimes X^x = G^{r(x)} \times \{x\}$. Thus condition (a) of Definition 1.22 is satisfied. Now given $f \in C_c(G \ltimes X)$ we have

$$\int_{G \ltimes X} f(\eta, y) d\mu^x(\eta, y) = \int_G f(\eta, x) d\lambda^{r(x)}(\eta)$$

and we would like to show that the function

$$x \mapsto \int_G f(\eta, x) d\lambda^{r(x)}(\eta) \tag{1.1}$$

is continuous.

Given $f \in C_c(G \ltimes X)$ we can extend f to a function in $C_c(G \times X)$, also denoted f, using Lemma 1.71. For $h \in C_0(G)$ and $g \in C_0(X)$ define $h \otimes g$ by $h \otimes g(\gamma, x) = h(\gamma)g(x)$. It follows from [RW98, Corollary B.17] that sums of functions of the form $h \otimes g$ are dense in $C_0(G \times X)$. Therefore, we can find sets $\{h_i^j\} \subset C_0(G)$ and $\{g_i^j\} \subset C_0(X)$ such that $k_i = \sum_j h_i^j \otimes g_i^j$ is a net and $k_i \to f$ uniformly. Let L_1 be the projection of $\operatorname{supp} f$ to G and let $g \in C_c(G)$ be one on L_1 and zero off some compact neighborhood of L_1. Similarly let L_2 be the projection of $\operatorname{supp} f$ to X and

let $h \in C_c(X)$ be one on L_2 and zero off some compact neighborhood of L_2. By replacing g_i^j with gg_i^j and h_i^j with hh_i^j we can assume, without loss of generality, that there exists a compact set K such that $\operatorname{supp}(k_i) \subset K$ for all i. Observe that for each i and j the function

$$x \mapsto g_i^j(x) \int_G h_i^j(\eta) d\lambda^{r(x)}(\eta) = \int_G h_i^j \otimes g_i^j(\eta, x) d\lambda^{r(x)}(\eta)$$

is continuous since it's built from continuous functions using composition and multiplication. Hence

$$x \mapsto \int_G k_i(\eta, x) d\lambda^{r(x)}(\eta)$$

is the sum of continuous functions and is continuous. Next, let K be the compact set given above and K' be its projection onto G. Since K' is compact it follows from Lemma 1.23 that $\{\lambda^u(K')\}$ is bounded by some number M. Thus given $y \in X$

$$\begin{aligned}\left|\int_G f(\eta, y) d\lambda^{r(y)}(\eta) - \int_G k_i(\eta, y) d\lambda^{r(y)}(\eta)\right| &= \left|\int_G (f - k_i)(\eta, y) d\lambda^{r(y)}(\eta)\right| \\ &\le \int_G |(f - k_i)(\eta, y)| d\lambda^{r(y)}(\eta) \\ &\le M\|f - k_i\|_\infty.\end{aligned}$$

Now suppose $x_j \to x$ is a net converging in X and $\epsilon > 0$. Choose i large enough so that $\|f - k_i\|_\infty < \epsilon/4M$ and J such that for all $j \ge J$ we have

$$\left|\int_G k_i(\eta, x_j) d\lambda^{r(x_j)}(\eta) - \int_G k_i(\eta, x) d\lambda^{r(x)}(\eta)\right| < \epsilon/2$$

Then for all $j \ge J$ we have

$$\begin{aligned}&\left|\int_G f(\eta, x_j) d\lambda^{r(x_j)}(\eta) - \int_G f(\eta, x) d\lambda^{r(x)}(\eta)\right| \\ &\qquad \le \left|\int_G f(\eta, x_j) d\lambda^{r(x_j)}(\eta) - \int_G k_i(\eta, x_j) d\lambda^{r(x_j)}(\eta)\right| \\ &\qquad\quad + \left|\int_G k_i(\eta, x_j) d\lambda^{r(x_j)}(\eta) - \int_G k_i(\eta, x) d\lambda^{r(x)}(\eta)\right| \\ &\qquad\quad + \left|\int_G f(\eta, x) d\lambda^{r(x)}(\eta) - \int_G k_i(\eta, x) d\lambda^{r(x)}(\eta)\right|\end{aligned}$$

$$\leq 2M\|f - k_i\|_\infty + \epsilon/2 < \epsilon$$

This proves that the function in (1.1) is continuous. The support of this function must be contained in $r(\operatorname{supp} f)$ and this implies that condition (b) of Definition 1.22 is satisfied.

Finally, suppose $(\gamma, x) \in G \ltimes X$ and $f \in C_c(G \ltimes X)$. Then

$$\begin{aligned}\int_{G\ltimes X} f((\gamma,x)(\eta,y))d\mu^{\gamma^{-1}\cdot x}(\eta,y) &= \int_G f((\gamma,x)(\eta,\gamma^{-1}\cdot x))\lambda^{r(\gamma^{-1}\cdot x)}(\eta)\\ &= \int_G f(\gamma\eta,x)\lambda^{s(\gamma)}(\eta)\\ &= \int_G f(\eta,x)\lambda^{r(\gamma)}(\eta)\\ &= \int_{G\ltimes X} f(\eta,y)\mu^x(\eta,y)\end{aligned}$$

This proves the left invariance condition and that $\{\mu^x\}$ is a Haar system for $G \ltimes X$. $\square$

Remark 1.73. When we view a group action of H on X as a groupoid action we can use Proposition 1.72 to form the transformation groupoid $H \ltimes X$. In this case $H \ltimes X$ is exactly the transformation group groupoid from Example 1.16 and the Haar system on $H \ltimes X$ is the one constructed in Example 1.27. Therefore Proposition 1.72 justifies the statements made in those two examples and we can use the phrase "transformation groupoid" without causing too much confusion.

Remark 1.74. In [Muh] the groupoid associated to a groupoid action is defined to be $G * X = \{(\gamma, x) \in G \times X : s(\gamma) = r(x)\}$ with the operations

$$(\gamma, \eta\cdot x)(\eta, x) = (\gamma\eta, x), \qquad\qquad (\gamma, x)^{-1} = (\gamma^{-1}, \gamma\cdot x).$$

It's easy to see that the map $\phi : G \ltimes X \to G * X$ defined by $\phi(\gamma, x) = (\gamma, \gamma^{-1}\cdot x)$ is an isomorphism between these two groupoids. The principal difference between them is that given $(\gamma, x) \in G \ltimes X$ we have $r(\gamma, x) = x$ and $s(\gamma, x) = \gamma^{-1}\cdot x$ and for $(\gamma, x) \in G * X$ we have $r(\gamma, x) = \gamma\cdot x$ and $s(\gamma, x) = x$. The reason that we are using the former groupoid is that it interacts nicely with crossed products.

Statements about groupoid actions often translate into similar statements about the transformation groupoid. For instance

Definition 1.75. Suppose G is a groupoid. We say G is *transitive* if given $u, v \in G^{(0)}$ there exists $\gamma \in G$ such that $s(\gamma) = u$ and $r(\gamma) = v$.

Remark 1.76. Note that a groupoid is transitive if and only if its associated orbit

groupoid is trivial. Also, in this case it is easy to see, using conjugation by elements in G, that all the stabilizer subgroups are isomorphic.

Definition 1.77. Suppose G is a groupoid and X is a G-space. We say X is *transitive* if given $x, y \in X$ there exists $\gamma \in G$ such that $\gamma \cdot x = y$. We say X is *orbit transitive* if given $x, y \in X$ such that $r(x)$ is orbit equivalent to $r(y)$ in $G^{(0)}$ then there exists $\gamma \in G$ such that $\gamma \cdot x = y$.

Remark 1.78. Using Proposition 1.59, it is straightforward to show that if X is a transitive G-space then G must be a transitive groupoid. If a non-transitive groupoid acts on X then the most that one could hope for is that X is orbit transitive.

Proposition 1.79. *Suppose G is a groupoid, X is a G-space and $G \ltimes X$ is the associated transformation groupoid. Then X is transitive if and only if $G \ltimes X$ is. Furthermore X is orbit transitive if and only if the range map on X factors to a bijection from X/G onto $G^{(0)}/G$.*

Proof. Well X is transitive if and only if given $x, y \in X$ we have $\gamma \in G$ such that $x = \gamma \cdot y$. However this occurs if and only if there exits $\gamma \in G$ such that

$$y = \gamma^{-1} \cdot x = s(\gamma, x), \quad \text{and} \quad x = r(\gamma, x).$$

Finishing the chain, this is equivalent to requiring that $G \ltimes X$ be transitive.

Next, $r : X \to G^{(0)}$ is always equivariant by Proposition 1.59 and as such factors to a surjection from X/G onto $G^{(0)}/G$. If X is orbit transitive then $G \cdot r(x) = G \cdot r(y)$ implies that $r(x)$ and $r(y)$ are orbit equivalent so that there exits $\gamma \in G$ such that $\gamma \cdot x = y$. Thus $G \cdot x = G \cdot y$ and r is injective. The reverse direction is just as simple. □

One of the important things about transformation groupoids is that the original groupoid action can be recovered from the action of the transformation groupoid on its unit space. What this will allow us to do is extend theorems about groupoids to theorems about groupoid actions with relatively little effort. The following proposition gives some indication of how this works because it will allow us to translate statements concerning the stabilizers and the orbit space of the action of G on X into statements concerning the stabilizers and orbit space of $G \ltimes X$.

Definition 1.80. Suppose G is a groupoid acting on a set X. Given $x \in X$ the *stabilizer subgroup* of G at x is

$$G_x := \{\gamma \in G : \gamma \cdot x = x\}$$

If G is a locally compact Hausdorff groupoid and X is a continuous G-space then we say the stabilizers *vary continuously* in G if $x_i \to x$ in X implies $G_{x_i} \to G_x$ in G with respect to the Fell topology.

Proposition 1.81. *Suppose G is a locally compact groupoid which acts continuously on the locally compact space X. Then G_x is a closed subgroup of the stabilizer subgroup $S_{r(x)}$ for all $x \in X$. Furthermore, G_x is naturally isomorphic to the stabilizer subgroup of $G \ltimes X$ at x and the stabilizers G_x vary continuously in G if and only if $G \ltimes X$ has continuously varying stabilizers. Finally, the orbit space X/G is naturally homeomorphic to $G \ltimes X^{(0)}/G \ltimes X$.*

Proof. If $\gamma \in G_x$ then $s(\gamma) = r(x)$ and $r(\gamma) = r(\gamma \cdot x) = r(x)$. Thus $G_x \subseteq S_{r(x)}$. Furthermore, it's straightforward to show that, because the action is continuous, G_x is closed. If $s, t \in G_x$ then $(st) \cdot x = s \cdot (t \cdot x) = x$ so that $st \in G_x$. Lastly if $s \in G_x$ then $s \cdot x = x$ so that, using Proposition 1.59, $x = s^{-1} \cdot x$. Thus G_x is a closed subgroup of $S_{r(x)}$. Next, let T_x the be the stabilizer subgroup of $G \ltimes X$ at x and define $\phi : G_x \to T_x$ by $\phi(t) = (t, x)$. It is clear that $r(t, x) = s(t, x) = x$ so that ϕ is well defined. What's more, ϕ is a homomorphism because

$$\phi(st) = (st, x) = (s, x)(t, x) = \phi(s)\phi(t).$$

Moving on, it's easy to see that ϕ is continuous. If we let $\psi : T_x \to G_x$ be defined by $\psi(t, x) = t$ then, since

$$t^{-1} \cdot x = s(t, x) = r(t, x) = x$$

we can conclude that t^{-1}, and hence t, are elements of G_x, making ψ well defined. Furthermore ψ is clearly a continuous inverse to ψ, making ψ an isomorphism of locally compact groups.

The following will make heavy use of Proposition 1.38. Suppose the stabilizers G_x vary continuously with respect to the Fell topology and suppose $x_i \to x$. Next, let $(s_i, x_i) \in T_{x_i}$ and suppose $(s_i, x_i) \to (s, x)$. Then, because the range and source maps are continuous and $x_i \to x$, we have $(s, x) \in T_x$. Now suppose $(s, x) \in T_x$. We know $s \in G_x$ so that we can pass to a subnet, relabel, and find $s_i \in G_{x_i}$ such that $s_i \to s$. Thus $(s_i, x_i) \in T_{x_i}$ and $(s_i, x_i) \to (s, x)$. It follows that $T_{x_i} \to T_x$ in the Fell topology. The opposite directly is proved in an entirely similar fashion.

Next, recall that we identify the unit space of $G \ltimes X$ with X via the map $(r(x), x) \leftrightarrow x$. In order to show that X/G and $X/G \ltimes X$ are naturally isomorphic it suffices to show that the actions induce the same orbit equivalence relation. However $x \sim y$ with respect to the action of G if and only if there exists $\gamma \in G$ such that $\gamma \cdot x = y$. This is true if and only if $x = \gamma^{-1} \cdot y$ which is true if and only if there exists $(\gamma, x) \in G \ltimes X$ such that $y = s(\gamma, x)$ and $x = r(\gamma, x)$. Continuing the string of implications, this is true if and only if $x \sim y$ with respect to the action of $G \ltimes X$. Since the actions induce the same orbit equivalence relation, the quotient of X with respect to each action is the same. $\square$

Remark 1.82. Proposition 1.81 helps answer another question. Given a continuous G-space X we would like to form a bundle out of the stabilizer groups G_x. Since Proposition 1.81 tells us that the stabilizers of the action are exactly the stabilizers of the transformation groupoid, it is natural to bundle the G_x together inside $G \ltimes X$. In other words, the stabilizer group bundle T for the action of G on X is nothing more than the stabilizer subgroupoid of $G \ltimes X$. Of course, once you work out all the definitions this just boils down to defining the stabilizer bundle to be the set

$$T := \{(s, x) \in G \times X : s \in G_x\}.$$

1.2.1 Groupoid Equivalence

One very important application of groupoid actions is the the notion of groupoid equivalence. This is a fundamental idea and will be a key tool in Section 6.1. The following material is taken, and expanded, from [MRW87].

Definition 1.83. Suppose G is a groupoid and X is a G-space. We say the action of G on X is *free* if $\gamma \cdot x = x$ implies $\gamma \in G^{(0)}$. If G is a locally compact Hausdorff groupoid and X is a continuous G-space then we say the action is *proper* if the map from $G * X$ to $X \times X$ given by $(\gamma, x) \mapsto (\gamma \cdot x, x)$ is proper.[6] The action is *principal*, and X is called a *principal G-space*, if it is both free and proper.

The following proposition gives a useful characterization of proper actions that helps explain why we are interested in them. The proof is lifted from [Wil07, Lemma 3.24].

Proposition 1.84. *Suppose G is a locally compact Hausdorff groupoid acting continuously on a locally compact Hausdorff space X. The action is proper if and only if given nets $\{x_i\}_{i\in I} \in X$ and $\{\gamma_i\}_{i\in I} \in G$ such that $x_i \to x$ and $\gamma_i \cdot x_i \to y$ then $\{\gamma_i\}$ has a convergent subnet.*

Proof. Suppose the action is proper so that the map $\phi : G * X \to X \times X$ given by $\phi(\gamma, x) = (\gamma \cdot x, x)$ is a proper map. Given elements $x_i, x, y \in X$ and $\gamma_i, \gamma \in G$ as in the statement of the proposition, let K be a compact neighborhood of x and y. We eventually have $x_i, \gamma_i \cdot x_i \in K$ so that eventually $(\gamma_i \cdot x_i, x_i) \in K \times K$. Hence, eventually, $(\gamma_i, x_i) \in \phi^{-1}(K \times K)$. Since ϕ is proper, $\phi^{-1}(K \times K)$ is compact and $\{\gamma_i\}$ must have a convergent subnet.

Now we will prove the reverse direction. Suppose K is a compact subset of $X \times X$ and $\{(\gamma_i, x_i)\}$ is a net in $\phi^{-1}(K)$. Then $\{(\gamma_i \cdot x_i, x_i)\}$ is a net in K so that we can pass to a subnet, relabel, and find $(y, x) \in K$ such that $\gamma_i \cdot x_i \to y$ and $x_i \to x$. However,

[6] A continuous map is proper if the inverse image of every compact set is compact.

we can now use our hypothesis to pass to a subnet, relabel, and find γ such that $\gamma_i \to \gamma$. Since the action is continuous $\gamma_i \cdot x_i \to \gamma \cdot x$ and we have $\gamma \cdot x = y$. Finally, $(\gamma_i, x_i) \to (\gamma, x)$ and $\phi(\gamma, x) = (y, x)$ so that $\{(\gamma_i, x_i)\}$ has a convergent subnet in $\phi^{-1}(K)$ and we are done. □

One good thing about proper actions of well behaved groupoids is that they have nice orbit spaces.

Proposition 1.85. *If G is a locally compact Hausdorff groupoid with open range and source maps and X is a proper G-space then the orbit space X/G is locally compact Hausdorff.*

Proof. We proved in Proposition 1.69 that the quotient map q is open and that X/G is locally compact. Suppose we have a net $G \cdot x_i$ in X/G such that $G \cdot x_i \to G \cdot x$ and $G \cdot x_i \to G \cdot y$. It will suffice to show $G \cdot x = G \cdot y$. Using the fact that q is open, we can pass to a subnet, relabel, and choose new representatives x_i so that $x_i \to x$. Then we pass to another subnet, relabel, and this time find $\gamma_i \in G$ such that $\gamma_i \cdot x_i \to y$. Since the action is proper we can use Proposition 1.84 to pass to yet another subnet, relabel, and find γ such that $\gamma_i \to \gamma$. It now follows from the continuity of the action that $\gamma_i \cdot x_i \to \gamma \cdot x$. Since X is Hausdorff, $\gamma \cdot x = y$ and $G \cdot x = G \cdot y$. □

Example 1.86. The most basic example of a principal G-space is the action from Example 1.64 where we let G act on itself by multiplication. It is easy enough to show that this action is free and proper.

Definition 1.87. Suppose G and H are locally compact Hausdorff groupoids which have open range and source maps. We say that a locally compact space Z is a (G, H)-*equivalence* if

- **(a)** Z is a left *strong* principal G-space,
- **(b)** Z is a right *strong* principal H-space,
- **(c)** the G and H actions commute,
- **(d)** the map s_X induces a bijection of Z/H onto $G^{(0)}$, and
- **(e)** the map r_X induces a bijection of $G\backslash Z$ onto $H^{(0)}$.

Remark 1.88. Recall that a G-space X is "strong" if the structure map from X to $G^{(0)}$ is open. This is a necessary condition when dealing with (G, H)-equivalences. Colloquially, when dealing with groupoid equivalence every range and source map in sight has to be open.

Remark 1.89. It is remarked in [MRW87] that (G, H)-equivalence induces an equivalence relation on locally compact groupoids. It is easy to see that the left and right action of G on G makes G a (G, G)-equivalence. Given a (G, H)-equivalence one can just swap the left and right actions to form a (H, G)-equivalence. Finally, given a (G, H)-equivalence Z and a (H, K)-equivalence Y we define $Z *_H Y$ to be the quotient of $Z * Y = \{(z, y) \in Z \times Y : s(z) = r(y)\}$ where we identify (z, y) with $(z \cdot \gamma, \gamma \cdot y)$ for all $z \in Z$, $y \in Y$ and $\gamma \in H$. It's not hard to see that G and K act naturally on $Z *_H Y$ and that $Z *_H Y$ is a (G, K)-equivalence.

The definition of a (G, H)-equivalence is a little complicated so one might expect that they are relatively rare. In fact, any strong principal G-space gives rise to an equivalence.

Definition 1.90. Suppose H is a locally compact Hausdorff groupoid with open range and source maps and X is a strong principal right H-space. Let H act on $X * X = \{(x, y) \in X \times X : s(x) = s(y)\}$ via the diagonal action

$$(x, y) \cdot \gamma := (x \cdot \gamma, y \cdot \gamma)$$

and define $X^H := (X * X)/H$. Denoting the image of (x, y) in X^H as $[x, y]$ we let $(X^H)^{(2)} = \{([x, y], [w, z]) \in X^H \times X^H : y = w\}$ and we define groupoid operations on X^H via

$$[x, y][y, z] = [x, z], \qquad\qquad [x, y]^{-1} = [y, x].$$

Then equipped with these operations, X^H is called the *imprimitivity groupoid* associated to X and H. The imprimitivity groupoid associated to a left action is defined analogously.

Remark 1.91. Propositions 1.92, 1.93 and 1.94 all have corresponding statements for imprimitivity groupoids built from strong principal *left* G-spaces and the proofs are all exactly the same. We will endeavor to always build imprimitivity groupoids from right actions, but there isn't any real difference one way or the other.

Of course, we made a number of claims in the definition that need to be verified.

Proposition 1.92. *Let H be a locally compact Hausdorff groupoid with open range and source maps and X a strong principal right H-space. Then the imprimitivity groupoid X^H is a locally compact Hausdorff groupoid which is second countable if X is. The unit space of X^H can be identified with X/H and under this identification $r([x, y]) = [x]$ and $s([x, y]) = [y]$. Furthermore, the range and source maps of X^H are open.*

Proof. First we show that X^H is a locally compact Hausdorff space. It is straightforward to show that the diagonal action of H on $X * X = \{(x, y) : s(x) = s(y)\}$ is

a continuous groupoid action. Using Proposition 1.69 we conclude that the quotient map $r : X * X \to X^H$ is open, $X^H = X * X/H$ is locally compact, and X^H is second countable if X, and hence $X * X$, is second countable. Next we will show that the action of H on $X * X$ is proper. Suppose $\{(x_i, y_i)\}$ is a net in $X * X$ and $\{\gamma_i\}$ is a net in H such that $(x_i, y_i) \to (x, y)$ and $(x_i \cdot \gamma_i, y_i \cdot \gamma_i) \to (w, z)$. Then, using Proposition 1.84, and the fact that the action of H on X is proper, we can find a convergent subnet of γ_i. Thus the action of H on $X * X$ is proper and X^H must be a Hausdorff space by Proposition 1.85.

We must spend some time showing that the operations in Definition 1.90 are well defined. For instance, if $[x, y] = [x', y']$ and $[y, z] = [y', z']$ then there exists $\gamma, \eta \in H$ such that $y = y' \cdot \gamma$ and $y = y' \cdot \eta$. However, it's easy to see that, because the action of H on X is free, we must have $\gamma = \eta$. Hence $(x, z) = (x', z') \cdot \gamma$ and multiplication is well defined. We can also do a similar calculation to show that the inverse is well defined. At this point it is straightforward to show that these operations make X^H into a groupoid. In order to see that the actions are continuous one must use the fact that the quotient map $Q : X * X \to X * X/H$ is open. For example, if $[x_i, y_i] \to [x, y]$ then by passing to a subnet, relabeling, and possibly choosing new representatives x_i and y_i, we can assume $x_i \to x$ and $y_i \to y$. However, it's now clear that $[y_i, x_i] \to [y, x]$ so that the inverse is continuous. Similar considerations show that the multiplication is also continuous and that X^H is a topological groupoid.

Now $r([x, y]) = [x, y][y, x] = [x, x]$ and similarly $s([x, y]) = [y, y]$. The map $\phi : X \to X*X$ such that $\phi(x) = (x, x)$ is a homeomorphism of X onto the diagonal in $X*X$. Since $\phi(x \cdot \gamma) = \phi(x) \cdot \gamma$, it is straightforward to show that this homeomorphism factors to a homeomorphism $\bar{\phi}$ from X/H onto $\{[x, x] \in X^H : x \in X\}$. Under this identification we clearly have $r([x, y]) = [x]$ and $s([x, y]) = [x]$. Finally, we show that the range and source maps are open. Suppose $[x_i] \to [x]$ and $r([x, y]) = [x]$. Since $q : X \to X/H$ is open we can pass to a subnet, relabel, and choose new representatives so that $x_i \to x$. Now $s(x_i) \to s(x)$ and $s(y) = s(x)$. Using the fact that the action of H on X is strongly continuous we can pass to another subnet, reindex, and find y_i in X such that $y_i \to y$ and $s(y_i) = s(x_i)$. Thus $(x_i, y_i) \in X * X$ and clearly $(x_i, y_i) \to (x, y)$. It follows that $[x_i, y_i] \to [x, y]$ and that $r([x_i, y_i]) = [x_i]$. Hence the range map is open on X^H. We could run through a similar argument to show that the source map is open, or we could observe that the source map is equal to the range map composed with the inverse map and that the inverse map is a homeomorphism. $\square$

The reason we care about the imprimitivity groupoid is that it turns out to be naturally equivalent to H.

Proposition 1.93. *Suppose H is a locally compact Hausdorff groupoid with open range and source maps and X is a strong principal right H-space. Then the imprimitivity groupoid X^H has a strong principal action on the left of X. The range map*

$r_X : X \to X/H$ is the quotient map and the action is defined by

$$[x, y] \cdot z = x \cdot \gamma$$

where γ is the unique element of H such that $z = y \cdot \gamma$. Furthermore, with these two actions X is a (X^H, H)-equivalence.

Proof. First, observe that r_X is open by Proposition 1.69. If $s([x, y]) = [x] = [z] = r(z)$ then there exists $\gamma \in H$ such that $y \cdot \gamma = z$. Furthermore if $\eta \in G$ such that $y \cdot \eta = z$ then $y \cdot \gamma\eta^{-1} = y$ so that $\gamma\eta^{-1} \in G^{(0)}$ and $\gamma = \eta$. Thus γ is unique and under our definition $[x, y] \cdot z = x \cdot \gamma$. Now, if $[x', y'] = [x, y]$ then there exists $\zeta \in G$ such that $(x, y) = (x', y') \cdot \zeta$. It follows that $y' \cdot \zeta\gamma = y \cdot \gamma = z$ so that $[x', y'] \cdot z = x' \cdot \zeta\gamma = x \cdot \gamma$. This shows that the action of X^H on X is well defined.

Next we will show it is a groupoid action. Suppose $z \in X$, and $[x, y], [y, w] \in X^H$ such that $[w] = [z]$. Let γ be the unique element such that $w \cdot \gamma = z$. Then $[y, w] \cdot z = y \cdot \gamma$. Clearly $[y \cdot \gamma] = [y]$ so that $[x, y]$ acts on $y \cdot \gamma$. The unique element of G we are looking for is again γ so that

$$[x, y] \cdot ([y, w] \cdot z) = [x, y] \cdot (y \cdot \gamma) = x \cdot \gamma.$$

On the other hand we clearly have $[x, w] \cdot z = x \cdot \gamma$ so that condition (a) of Definition 1.55 is satisfied. Since $x \cdot s(x) = x$ we have $[x, x] \cdot x = x$ for all $x \in X$ and condition (b) is satisfied as well. Now suppose $z_i \to z$ in X and $[x_i, y_i] \to [x, y]$ in X^H such that $[y_i] = [z_i]$ for all i. First, pass to a subnet. We will show that there is a sub-subnet such that $[x_i, y_i] \cdot z_i \to [x, y] \cdot z$. Now, pass to a subnet, relabel, and choose new representatives so that $x_i \to x$ and $y_i \to y$. For each i let γ_i be such that $y_i \cdot \gamma_i = z_i$ and let γ be such that $y \cdot \gamma = z$. Using the fact that the action of G is proper we can conclude that by passing to a subnet we can find η such that $\gamma_i \to \eta$. However, we then have $y \cdot \eta = z$ so that $\eta = \gamma$ by freeness. We can use this trick to show that every subnet of γ_i has a subnet which converges to γ. This implies $\gamma_i \to \gamma$. Hence $x_i \cdot \gamma_i \to x \cdot \gamma$, but this is exactly what we needed to show.

We have shown that X is a strong left X^H-space. We will now show that it is principal. Suppose $[x, y] \cdot z = z$. Let γ be such that $y \cdot \gamma = z$. Since $[x, y] \cdot z = x \cdot \gamma = z$ we can conclude that $y = z \cdot \gamma^{-1} = x$ and $[x, y] \in (X^H)^{(0)}$. Thus the action is free. Now suppose we have $z_i, w, z \in X$ and $[x_i, y_i] \in X^H$ such that $z_i \to z$ and $[x_i, y_i] \cdot z_i \to w$. Choose $\gamma_i \in G$ so that $y_i \cdot \gamma_i = z_i$ and $[x_i, y_i] \cdot z_i = x_i \cdot \gamma_i$. Now we have $x_i \cdot \gamma_i \to w$ and $y_i \cdot \gamma_i \to z$. However, this implies $(x_i, y_i) \cdot \gamma_i \to (w, z)$ in $X * X$ and that $[x_i, y_i] \to [w, z]$ in X^H. Thus the action is proper and we are done.

Now we will show that X is a (X^H, H)-equivalence. We have already shown, or assumed, that H and X^H have open range and source maps, that X is a strong principal right H-space, and that X is a strong principal left X^H-space. Furthermore we have already seen that the range map on X is nothing more than the quotient

map from X to $X/H \cong (X^H)^{(0)}$ which clearly factors to a bijection of X/H onto itself. Now consider the source map $s : X \to G^{(0)}$. All that we need to do to show s factors to a bijection is show that if $s(x) = s(y)$ then there exists $[w, z] \in X^H$ such that $[w, z] \cdot x = y$. However if $s(x) = s(y)$ then $[x, y] \in X^H$ and it's easy to see that $[y, x] \cdot x = y$. All that's left is to show that the actions commute. Well, suppose $\gamma \in G$, $z \in X$ and $[x, y] \in X^H$ with the appropriate ranges and sources. Let $\eta \in G$ be such that $y \cdot \eta = z$. Then $([x, y] \cdot z) \cdot \gamma = x \cdot \eta\gamma$. Since $y \cdot \eta\gamma = z \cdot \gamma$ it follows that $[x, y] \cdot (z \cdot \gamma) = x \cdot \eta\gamma$. Thus the actions commute and X is a (X^H, H)-equivalence. $\square$

It turns out that this situation is the most general one.

Proposition 1.94. *Suppose G and H are locally compact Hausdorff groupoids and Z is a (G, H)-equivalence. There is a natural isomorphism ϕ between G and the imprimitivity groupoid Z^H defined as follows. Given $[x, y] \in Z^H$ we have $s(x) = s(y)$ so that there exists a unique $\gamma \in G$ such that $x = \gamma \cdot y$ and we define $\phi([x, y]) := \gamma$.*

Proof. First we show that ϕ is well defined. Suppose $[x, y] = [x', y']$. Then there exists $\eta \in H$ such that $x \cdot \eta = x'$ and $y \cdot \eta = y'$. Let γ be the unique element of G such that $x = \gamma \cdot y$. Since the actions commute $x \cdot \eta = (\gamma \cdot y) \cdot \eta = \gamma \cdot (y \cdot \eta)$. Therefore $\phi([x, y]) = \gamma = \phi([x', y'])$. Next we will show that ϕ is a homomorphism. Now suppose $[x, y], [y, z] \in Z^H$ and let $\gamma, \eta \in G$ such that $x = \gamma \cdot y$ and $y = \eta \cdot z$. Then $s(\gamma) = r(y) = r(\eta \cdot z) = r(\eta)$ and $x = \gamma \cdot y = \gamma\eta \cdot z$ so that $\phi([x, z]) = \gamma\eta = \phi([x, y])\phi([y, z])$.

Now suppose $[x_i, y_i] \to [x, y]$, $\phi([x_i, y_i]) = \gamma_i$, and $\phi([x, y]) = \gamma$. Pass to a subnet. We will show that there is a sub-subnet such that $\gamma_i \to \gamma$. Use the fact that the quotient map $Z * Z \to Z^H$ is open to pass to a subnet, reindex, and possibly choose new representatives x_i and y_i so that $x_i \to x$ and $y_i \to y$. Since $x_i = \gamma_i \cdot y_i$ for all i the fact that the action of G is proper implies that we can pass to a subnet and assume $\gamma_i \to \eta$. The continuity of the actions now implies that $\gamma_i \cdot y_i \to \eta \cdot y$. Since Z is Hausdorff we have $\eta \cdot y = \gamma \cdot y$ and freeness implies $\eta = \gamma$ and we are done.

Now suppose $\gamma \in G$. Then $s(\gamma) \in G^{(0)}$ we can choose an $x \in Z$ such that $r(x) = s(\gamma)$. Furthermore, since the range map on Z factors to a bijection, if $y \in Z$ such that $r(y) = s(\gamma)$ then there exists $\eta \in H$ such that $x = y \cdot \eta$. Now define $\psi : G \to Z^H$ by $\psi(\gamma) = [x, \gamma \cdot x]$. We need to show ψ is well defined. Given $\eta \in H$ we have $(x, \gamma \cdot x) \cdot \eta = (x \cdot \eta, \gamma \cdot (x \cdot \eta))$ and therefore $[x, \gamma \cdot x] = [x \cdot \eta, \gamma \cdot (x \cdot \eta)]$. Thus $\psi(\gamma)$ is independent of the choice of x. Next, it's clear that $\phi(\psi(\gamma)) = \phi([x, \gamma \cdot x]) = \gamma$. Furthermore, if $\phi([x, y]) = \gamma$ then $y = \gamma \cdot x$ so that $\psi(\phi([x, y])) = \psi(\gamma) = [x, \gamma \cdot x] = [x, y]$. Therefore ϕ and ψ are inverses and ϕ is a bijection. The last thing we need to show is that ψ is continuous. Suppose $\gamma_i \to \gamma$. Pass to a subnet. Our goal is to show a sub-subnet of $\psi(\gamma_i)$ converges to $\psi(\gamma)$. Well $s(\gamma_i) \to s(\gamma)$ and we can use the fact that the range map on Z is open to pass to a subnet and find $x_i \to x$ such that

$s(x_i) = r(\gamma_i)$. It follows that $\gamma_i \cdot x_i \to \gamma \cdot x$ and $[x_i, \gamma_i \cdot x_i] \to [x, \gamma \cdot x]$. Thus ψ is continuous and ϕ is an isomorphism of locally compact Hausdorff groupoids. $\square$

The following proposition describes the imprimitivity groupoid that will be of interest in Chapter 6. The principal action is exactly the one described in Example 1.63.

Proposition 1.95. *Suppose G is a locally compact Hausdorff groupoid with a Haar system and that H is a closed subgroupoid such that the range and source maps restricted to H are open. In particular, this is true if H has its own Haar system. If $X = s^{-1}(H^{(0)})$ then H acts on the right of X via multiplication and with this action X is a strong principal H-space. The associated imprimitivity groupoid, in this case denoted G^H, has a Haar system $\{\mu^{[\xi]}\}$ defined for $f \in C_c(G^H)$ by*

$$\int_{G^H} f([\xi,\eta])d\mu^{[\zeta]}([\xi,\eta]) = \int_G f([\zeta,\eta])d\lambda_{s(\zeta)}(\eta). \tag{1.2}$$

Proof. The action of H on X is exactly the action described in Example 1.63 and it was shown there that if $s : X \to H^{(0)}$ is the restriction of the source map to X and $\xi \cdot \eta = \xi\eta$ then X is a strong right H-space. If $\xi \cdot \eta = \xi\eta = \xi$ for $\xi \in X$ and $\eta \in H$ then we can multiply both sides by ξ^{-1} and get $\eta = \xi^{-1}\xi = s(\xi) \in H^{(0)}$. Thus the action of H on X is free. Now suppose $\xi_i \to \xi$ and $\xi_i \cdot \eta_i \to \zeta$. Using the fact that the groupoid operations are continuous we have

$$\xi_i^{-1}(\xi_i\eta_i) = \eta_i \to \xi^{-1}\zeta.$$

Since H is closed, $\xi^{-1}\zeta \in H$ and η_i converges in H. Thus the action of H on X is proper and therefore principal. Since H has open range and source maps, by assumption, and X is a strong principal right H-space, we can use Proposition 1.93 to construct the imprimitivity groupoid G^H.

We would like to show that the $\mu^{[\xi]}$ defined in the statement of the proposition form a Haar system for G^H. This is actually proved in [IW09] and has been expanded here for reference. First we show that (1.2) is well defined. Suppose $\gamma \in H$ such that $r(\gamma) = s(\zeta)$. Then, noting that applying left invariance to $\lambda_u = (\lambda^u)^{-1}$ gives us "right invariance", we get

$$\int_G f([\zeta\gamma,\eta])d\lambda_{s(\gamma)}(\eta) = \int_G f([\zeta\gamma,\eta\gamma])d\lambda_{s(\zeta)}(\eta) = \int_G f([\zeta,\eta])d\lambda_{s(\zeta)}(\eta).$$

Thus (1.2) is independent of the representative ζ. It is clear that this defines a positive linear functional on $C_c(G^H)$ so that $\mu^{[\zeta]}$ is a non-negative Radon measure for all $[\zeta] \in G^H$. Now suppose $[\zeta,\gamma] \in (G^H)^{[\zeta]}$ and U is an open neighborhood of $[\zeta,\gamma]$.

We may as well assume U is relatively compact and choose $f \in C_c(G^H)^+$ such that, $0 \le f \le 1$, f is one on $[\zeta, \gamma]$, and f is zero off U. It follows that

$$\mu^{[\zeta]}(U) \ge \int_{G^H} f([\xi,\eta])\,d\mu^{[\zeta]}([\xi,\eta]) = \int_G f([\zeta,\eta])\,d\lambda_{s(\zeta)}(\eta) > 0.$$

Now if $[\xi, \gamma] \notin G^{[\zeta]}$ we can pick a relatively compact open neighborhood U of $[\xi, \gamma]$ which is disjoint from $(G^H)^{[\zeta]}$. Since U is relatively compact we can choose another relatively compact neighborhood V such that $U \subset V$ and V is disjoint from $(G^H)^{[\zeta]}$ as well. Choose $f \in C_c(G^H)^+$ so that, $0 \le f \le 1$, f is one on U, and f is zero off V. Then

$$\mu^{[\zeta]}(U) \le \int_{G^H} f([\xi,\eta])\,d\mu^{[\zeta]}([\xi,\eta]) = \int_G f([\zeta,\eta])\,d\lambda_{s(\zeta)}(\eta) = 0$$

where the last equality holds because f is supported off $(G^H)^{[\zeta]}$. It follows that $\operatorname{supp}\mu^{[\zeta]} = (G^H)^{[\zeta]}$.

Next we need to prove the continuity condition. We start by showing that given $f \in C_c(X * X)$ the function

$$\gamma \mapsto \int_G f(\gamma,\eta)\,d\lambda_{s(\gamma)}(\eta) \tag{1.3}$$

is continuous on X. Use Lemma 1.71 to extend f to $C_c(X \times X)$. Just as in the proof of Proposition 1.72 we find a net of sums of elementary tensors $k_i = \sum_j g_i^j \otimes h_i^j$ which converge to f uniformly and are all supported in some compact set. We can then use the fact that

$$\gamma \mapsto \int_G g_i^j \otimes h_i^j(\gamma,\eta)\,d\lambda_{s(\gamma)}(\eta) = g_i^j(\gamma)\int_G h_i^j(\eta)\,d\lambda_{s(\gamma)}(\eta)$$

is clearly a continuous function to prove that (1.3) is continuous. The computation is exactly the same as in the proof of Proposition 1.72 and won't be reproduced here. Next, we need to show that the "factorization" of (1.3) is continuous. This result is proved in [Ren87, Lemme 1.3] but that paper is in French so the proof is included for reference. Define

$$Y = G^H * X := \{([\gamma,\eta],\zeta) \in G^H \times X : [\gamma] = [\zeta]\}.$$

and let $\phi : X * X \to Y$ be given by $\phi(\gamma,\eta) = ([\gamma,\eta],\gamma)$. We claim that ϕ is a homeomorphism. It is clear that ϕ is continuous and it is straightforward to show that ϕ is bijective. In particular, the inverse of $([\gamma,\eta],\zeta)$ is $(\zeta, \eta\delta)$ where δ is the unique element of H such that $\zeta = \gamma\delta$. We will restrict ourselves to showing that it has a continuous inverse. Suppose $([\gamma_i,\eta_i],\zeta_i) \to ([\gamma,\eta],\zeta)$ and let δ_i and δ be as

above. Pass to a subnet. It will suffice to show that there is a sub-subnet such that $\eta_i\delta_i \to \eta\delta$. By passing to a subnet we may assume that $\eta_i \to \eta$ and $\gamma_i \to \gamma$. However $\zeta_i = \gamma_i\delta_i$ and $\zeta = \gamma\delta$ so that we can use the fact that the action of H is principal to pass to another subnet and assume $\delta_i \to \delta$. The result follows.

Next, observe that given $f \in C_c(Y)$ we have

$$\begin{aligned}\int_G f(\phi(\zeta,\eta))d\lambda_{s(\zeta)}(\eta) &= \int_G f([\zeta,\eta],\zeta)d\lambda_{s(\zeta)}(\eta)\\ &= \int_Y f([\gamma,\eta],\xi)d(\mu^{[\zeta]}\times\delta_\zeta)([\gamma,\eta],\xi).\end{aligned}$$

In other words ϕ identifies the measures $\lambda_{s(\gamma)}$ and $\mu^{[\zeta]}\times\delta_\zeta$. Fix $f \in C_c(G^H)$ and suppose $g \in C_c(X)$. Define $F \in C_c(Y)$ by $F = f\otimes g$ so that $F([\gamma,\eta],\zeta) = f([\gamma,\eta])g(\zeta)$ and observe that

$$\zeta \mapsto \int_Y F([\gamma,\eta],\xi)d(\mu^{[\zeta]}\times\delta_\zeta)([\gamma,\eta],\xi) = \int_G F(\phi(\zeta,\eta))d\lambda_{s(\zeta)}(\eta)$$

is continuous since (1.3) is continuous. However this implies that

$$\zeta \mapsto \int_Y F([\gamma,\eta],\xi)d(\mu^{[\zeta]}\times\delta_\zeta)([\gamma,\eta],\xi) = g(\zeta)\int_{G^H} f([\gamma,\eta])d\mu^{[\zeta]}([\gamma,\eta]) \tag{1.4}$$

is continuous. Suppose $[\zeta_i] \to [\zeta]$ and, by possibly passing to a subnet and choosing new representatives, assume that $\zeta_i \to \zeta$. We can, without loss of generality suppose that this sequence converges inside some compact neighborhood K. Choose $g \in C_c(X)$ so that g is one on K. Then the fact that (1.4) is continuous implies that

$$\int_{G^H} f([\gamma,\eta])d\mu^{[\zeta_i]}([\gamma,\eta]) \to \int_{G^H} f([\gamma,\eta])d\mu^{[\zeta]}([\gamma,\eta]).$$

This is enough to show that $[\zeta] \mapsto \int_{G^H} fd\mu^{[\zeta]}$ is continuous and it's easy to see that this function is compactly supported.

All we have to do now is verify the left invariance condition. Suppose $[\gamma,\eta] \in G^H$ and $f \in C_c(G^H)$. Then

$$\begin{aligned}\int_{G^H} f([\gamma,\eta][\zeta,\xi])d\mu^{[\eta]}([\zeta,\xi]) &= \int_G f([\gamma,\eta][\eta,\xi])d\lambda_{s(\eta)}(\xi) = \int_G f([\gamma,\xi])d\lambda_{s(\eta)}(\xi)\\ &= \int_G f([\gamma,\xi])d\lambda_{s(\gamma)}(\xi) \quad \text{since } s(\gamma)=s(\eta)\\ &= \int_{G^H} f([\zeta,\xi])d\mu^{[\gamma]}([\zeta,\xi]).\end{aligned}$$

Thus $\mu^{[\zeta]}$ is left invariant and the collection $\{\mu^{[\zeta]}\}$ forms a Haar system for G^H. $\square$

Remark 1.96. Observe that in Proposition 1.95 we did not have to assume that H has a Haar system for G^H to have one, only that it has open range and source maps. It seems to be an open, and difficult question, to ask if there are two equivalent groupoids such that one has a Haar system and the other doesn't.

1.3 Amenable Groupoids

Groupoid amenability will only play a role as an assumption in one of the major results of the thesis, and will never be used in a technical way. However, since groupoid amenability is a very new subject, it seems appropriate to include some of the basic definitions and theorems. We will use the notion of amenability that Anantharaman-Delaroche and Renault use in [ADR00].

Definition 1.97. Suppose X and Y are locally compact Hausdorff spaces and $\pi : X \to Y$ is a continuous surjection. A continuous system of measures for π is a set of positive Radon measures $\alpha = \{\alpha^y\}_{y\in Y}$ on X such that

(a) $\operatorname{supp}\alpha^y \subseteq \pi^{-1}(y)$,

(b) and for every $f \in C_c(X)$ the function $\alpha(f) : Y \to \mathbb{C}$ such that

$$\alpha(f)(y) := \int_X f d\alpha^y$$

is continuous and compactly supported.

We say α is *proper* if $\alpha^y \neq 0$ for all $y \in Y$ and we say α is *full* if $\operatorname{supp}\alpha^y = \pi^{-1}(y)$ for all $y \in Y$.

Definition 1.98. Let G be a locally compact groupoid, X and Y locally compact strong G-spaces, and $\pi : X \to Y$ a continuous G-equivariant surjection. An *invariant continuous* π-system is a continuous system of measures α for π such that given $(\gamma, y) \in G * Y$ we have $\gamma\alpha^y = \alpha^{\gamma\cdot y}$. Here $\gamma\alpha^y$ denotes the measure defined on $C_c(X)$ via

$$\gamma\alpha^y(f) = \int_X f(\gamma \cdot x) d\alpha^y(x).$$

Example 1.99. Given a locally compact groupoid G with Haar system λ, it is clear that λ is a full, invariant, continuous r-system. In fact, "full, invariant, continuous r-system" is just a very short, alliterative way to define a Haar system.

Definition 1.100. Let G be a locally compact groupoid, X and Y locally compact strong G-spaces, and $\pi : X \to Y$ a continuous G-equivariant surjection. An *approximate continuous invariant mean* (a.c.i.m.) for π is a net $\{m_i\}$ of continuous systems of probability measures for π such that $\|\gamma m_i^y - m_i^{\gamma\cdot y}\|_1 \to 0$ uniformly on compact subsets of $G * Y$.[7]

It may be helpful to observe that the invariance condition for an invariant π-system α can be written $\|\gamma\alpha^y - \alpha^{\gamma\cdot y}\|_1 = 0$. In this light, an a.c.i.m. is a net of systems of *probability* measures that, in the limit, behave like an invariant system. Next, we define amenability for maps, groupoids and G-spaces. It's notable that amenability for groupoid actions is defined in terms of the transformation groupoid.

Definition 1.101. We say a continuous G-equivariant surjection $\pi : X \to Y$ between strong G-spaces X and Y is *amenable* if it admits an approximate continuous invariant mean. In particular, we say that a locally compact Hausdorff groupoid with open range and source maps is (topologically) amenable if the range map $r : G \to G^{(0)}$ is amenable, where we view G and $G^{(0)}$ as G-spaces in the usual way. We say that a locally compact G-space X is amenable if the groupoid $G \ltimes X$ is.

Remark 1.102. Suppose G is a locally compact Hausdorff groupoid with open range and source maps. It's clear that the action of G on itself is strongly continuous. Furthermore, the action of G on $G^{(0)}$ is also strongly continuous. Finally, given γ, η in G such that $s(\gamma) = r(\eta)$ we have

$$r(\gamma\eta) = r(\gamma) = \gamma \cdot r(\eta)$$

so that r is a continuous G-equivariant surjection. Thus, Definition 1.101 makes sense.

Example 1.103. Suppose G is a locally compact Hausdorff groupoid which admits a Haar system $\{m^u\}$ of probability measures. Then m is, by definition, a continuous invariant r-system. We can form an a.c.i.m. using the constant sequence $m_i^u = m^u$ for all i. In some sense, amenability is meant to be a generalization of this situation.

Next we give a couple of useful characterizations of amenability. As usual, we have to start out with a definition.

Definition 1.104. Given a continuous surjection $\pi : X \to Y$ between locally compact Hausdorff spaces a set $A \subset X$ will be called *π-compact* if, for every compact $K \subset Y$, its intersection with $\pi^{-1}(K)$ is compact. We will use $C_{c,\pi}(X)$ to denote the space of continuous functions on X with π-compact support. When G is a locally compact Hausdorff groupoid and $r : G \to G^{(0)}$ is the range map then r-compact sets will be called *conditionally compact*.

[7]For a finite measure m on Y we define $\|m\|_1 := |m|(Y)$.

Definition 1.105. Suppose the locally compact Hausdorff groupoid G acts on a locally compact Hausdorff space X. A function e on $G \ltimes X$ is said to be of *positive type* if, for every $x \in X$, every $n \in \mathbb{N}$, and every $\gamma_1, \ldots, \gamma_n \in G^{r(x)}$, and $z_1, \ldots, z_n \in \mathbb{C}$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e(\gamma_i^{-1}\gamma_j, \gamma_i^{-1} \cdot x)\bar{z}_i z_j \geq 0.$$

We denote by $e^{(0)}$ the restriction of e to $G^{(0)} \ltimes X$

The following is a restatement of [ADR00, Propositions 2.2.5,2.2.6].

Proposition 1.106. *Suppose G is a locally compact Hausdorff groupoid and let G have a strongly continuous action on X and Y. Now suppose we have a continuous G-equivariant surjection $\pi : X \to Y$ such that there exists an invariant continuous π-system α of measures. Then properties* (b),(c) *and* (d) *below are all equivalent and each of them implies* (a). *Furthermore, if X is a proper G-space then the following are equivalent.*

(a) *π is amenable.*

(b) *There exists a net of positive functions $\{g_i\} \subset C_{c,\pi}(X)$ such that*

(i) *$\int g_i d\alpha^y = 1$ for all $y \in Y$ and*

(ii) *$\int |g_i(\gamma \cdot x) - g_i(x)| d\alpha^y(x)$ converges to zero uniformly on the compact subsets of $G * Y$.*

(c) *There exists a net $\{\xi_i\} \subset C_{c,\pi}(X)$ such that*

(i) *$\int |\xi_i|^2 d\alpha^y = 1$ for all $y \in Y$*

(ii) *$\int |\xi_i(\gamma \cdot x) - \xi_i(x)|^2 d\alpha^y(x)$ converges to zero uniformly on the compact subsets of $G * Y$.*

(d) *There exists a net $\{\xi_i\} \subset C_{c,\pi}(X)$ such that the net $\{e_i\}$ of positive type functions on the groupoid $G \ltimes Y$ defined by*

$$e_i(\gamma, y) = \int_X \overline{\xi_i(x)} \xi_i(\gamma^{-1} \cdot x) d\alpha^y(x)$$

satisfies:

(i) *$e_i^{(0)} = 1$ for all i,*

(ii) *$\lim_i e_i = 1$ uniformly on compact subsets of $G \ltimes Y$.*

Remark 1.107. Proposition 1.106 is especially useful in the case of groupoid amenability. If the locally compact Hausdorff groupoid G has a Haar system then, by definition, there exists a continuous invariant r-system. Furthermore, the action of G on itself is always proper. Thus, in this situation, conditions (a)–(d) in Proposition 1.106 are all equivalent.

There are three more results in [ADR00] that we will state here without proof. In [ADR00] they are Propositions 2.2.13, 5.1.1, and 5.1.2 respectively.

Proposition 1.108. *For locally compact Hausdorff groupoids, amenability is invariant under groupoid equivalence.*

Proposition 1.109. *A locally closed subgroupoid H of an amenable locally compact Hausdorff groupoid G is amenable.*[8] *In particular, the stabilizer subgroupoid and all of the stabilizers subgroups are amenable if G is.*

Proposition 1.110. *Let G and H be locally compact Hausdorff groupoids with open range and source maps and $\pi : G \to H$ a continuous open surjective homomorphism such that $\pi^{(0)} : G^{(0)} \to H^{(0)}$ is a homeomorphism. We denote by $N = \{\gamma \in G : \pi(\gamma) \in H^{(0)}\}$ the kernel of π. Then G is amenable if and only if N and H are both amenable.*

Corollary 1.111. *Let G be a locally compact Hausdorff groupoid with open range and source and continuously varying stabilizer. Then G is amenable if and only if both the stabilizer subgroupoid S and the orbit groupoid R_Q are.*

Proof. Let $\pi : G \to R_Q$ be the canonical homomorphism. Since the stabilizers vary continuously we know from Proposition 1.53 that π is a continuous, surjective, open homomorphism and that R_Q is a locally compact Hausdorff groupoid with open range and source. Furthermore, π clearly restricts to a homeomorphism of $G^{(0)}$ with $R_Q^{(0)}$. (We usually identify $R_Q^{(0)}$ with $G^{(0)}$ via this map.) It is clear that the kernel of π is the stabilizer subgroupoid S. It follows from Proposition 1.110 that G is amenable if and only if both R_Q and S are. $\square$

Remark 1.112. We have given the briefest of introductions to topological amenability. None of the proofs of the above theorems are particularly difficult and, as stated before, can all be found in [ADR00]. There is also a measure theoretic notion of amenability. In [ADR00] Anantharaman-Delaroche and Renault go into extensive detail about measurable amenability and the properties of amenable measured groupoids. They also go into some depth about the relationship between amenability and the operator algebras associated to groupoids.

[8] A set is locally closed if it is the intersection of a closed set and an open set. In particular, every closed set is locally closed.

Chapter 2

Groupoid Group Bundles

This chapter contains two results concerning abelian group bundles. The first is the definition of a principal S-bundle where S is an abelian group bundle. We construct a sheaf cohomology theory for S and show that the isomorphism classes of principal S-bundles are in one-to-one correspondence with elements of the first cohomology group. We also connect this material back to existing work done for "locally σ-trivial" bundles. In Section 2.2 we prove a Pontryagin duality theorem for continuously varying, abelian group bundles and in Section 2.3 we present an interesting counterexample which shows that not every continuous bijective homomorphism between second countable, continuously varying, abelian group bundles is an isomorphism. This material is (mostly) low level and has been included before Chapter 3 to emphasize that.

2.1 Principal Group Bundles

The goal of this section is to generalize principal group bundles to situations where the bundle may not be locally trivial. In particular, we would like to be able to deal with group bundles as defined in Definition 1.34. However, as in the group case, commutativity will be an essential assumption.

Remark 2.1. For the rest of this section S will always denote an abelian locally compact Hausdorff group bundle with bundle map p. We will endeavor to state these hypothesis on all of the important theorems.

We will develop a theory of principal S-bundles that mirrors the classic theory of principal H-bundles where H is a locally compact Hausdorff group. As in the group case, there is a nice one-to-one correspondence between S-bundles and principal S-spaces with local sections. However, in order to be consistent we must start with the bundle definition and will develop the correspondence later. The material in this

section is modeled off [RW98, Section 4.2]. This first definition is really a matter of notation and terminology. Any surjection can be viewed as a bundle map, although it's not always useful to do so.

Definition 2.2. Suppose Y and X are topological spaces and $q : X \to Y$ is a continuous surjection. Then X is called a *(topological) bundle* over Y with bundle map q and the fibres $q^{-1}(y)$ are denoted by X_y for all $y \in Y$.

Principal S-bundles are bundles which are locally isomorphic to S. When dealing with "local triviality" conditions one must consider the fact that there may be different trivializations for the same bundle. Our first definition of principal bundle will depend on the trivialization.

Definition 2.3. Let S be an abelian locally compact Hausdorff group bundle with bundle map p. Suppose X is a locally compact Hausdorff bundle over $S^{(0)}$ with bundle map q. Furthermore, suppose there is an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of $S^{(0)}$ such that for each $i \in I$ there is a homeomorphism $\phi_i : q^{-1}(U_i) \to p^{-1}(U_i)$ with $p \circ \phi_i = q$. Finally, suppose that for all $i, j \in I$ there is a section γ_{ij} of $S|_{U_i \cap U_j} = p^{-1}(U_i \cap U_j)$ such that

$$\phi_i \circ \phi_j^{-1}(s) = \gamma_{ij}(p(s))s$$

for all $s \in S|_{U_i \cap U_j}$. Such a bundle is called a *principal S-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$.* The maps $\phi = \{\phi_i\}$ are referred to as *trivializing maps* and the sections $\gamma = \{\gamma_{ij}\}$ are referred to as *transition maps.*

Remark 2.4. If $\mathcal{U} = \{U_i\}$ is an open cover then we use the usual notation $U_{ij} := U_i \cap U_j$.

We would like to see that there is some freedom with respect to the neighborhoods in a trivialization. In particular, we would like to be able to refine them at will.

Proposition 2.5. *If X is a principal S-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$ then for any refinement $\mathcal{V}$ of the open cover $\mathcal{U}$ we can make X a principal S-bundle with trivialization $(\mathcal{V}, \phi_{\mathcal{V}}, \gamma_{\mathcal{V}})$ where $\phi_{\mathcal{V}}$ and $\gamma_{\mathcal{V}}$ denote the natural restriction of the trivializing and transition maps to $\mathcal{V}$.*

Proof. Suppose $\mathcal{V} = \{V_j\}_{j\in J}$ is a refinement of $\mathcal{U} = \{U_i\}_{i\in I}$ with refining map $r : I \to J$ so that $V_j \subseteq U_{r(j)}$ for all $j \in J$. We can define trivializing maps on V_j by $\phi'_j = \phi_{r(j)}|_{V_j}$ for all $j \in J$ and transition maps $\gamma'_{ij} = \gamma_{r(i)r(j)}|_{V_{ij}}$ for all $i, j \in J$. It is easy to see that ϕ'_i is a homeomorphism and that $p \circ \phi'_i = q$; all that you have to check is that ϕ'_i is surjective, which follows from the fact that ϕ preserves fibres. Furthermore

$$\phi'_i \circ (\phi'_j)^{-1}(s) = \phi_{r(i)} \circ \phi_{r(j)}^{-1}(s) = \gamma_{r(i)r(j)}(p(s))s = \gamma'_{ij}(p(s))s.$$

Thus the maps $\phi_{\mathcal{V}} = \{\phi'_j\}$ and $\gamma_{\mathcal{V}} = \{\gamma_{ij}\}$ make X into a principal S-bundle. ∎

Remark 2.6. We will usually drop the $\mathcal{V}$ from $\phi_{\mathcal{V}}$ and $\gamma_{\mathcal{V}}$ to avoid notational clutter.

Now we define the notion of an S-bundle isomorphism. The basic idea is that an isomorphism is locally given by sections.

Definition 2.7. Suppose $q : X \to S^{(0)}$ and $r : Y \to S^{(0)}$ are both principal S-bundles with trivializations $(\mathcal{U}, \phi, \gamma)$ and $(\mathcal{V}, \psi, \eta)$ respectively. Let $\mathcal{W}$ be some common refinement of $\mathcal{U}$ and $\mathcal{V}$ with refining maps σ and ρ respectively. Furthermore, suppose $\Omega : X \to Y$ is a homeomorphism such that $r \circ \Omega = q$ and that for all $W_i \in \mathcal{W}$ $\beta_i : W_i \to S$ is a section of p such that for all $s \in p^{-1}(W_i)$

$$\psi_{\sigma(i)} \circ \Omega \circ \phi_{\rho(i)}^{-1}(s) = \beta_i(p(s))s.$$

Then $(\mathcal{W}, \Omega, \beta)$ is an S-bundle *isomorphism* of X onto Y.

A special case of a principal S-bundle isomorphism is when some principal S-bundle X comes with two trivializations which "agree" on a common refinement.

Definition 2.8. Suppose X is a principal S-bundle and that $(\mathcal{U}, \phi, \gamma)$ and $(\mathcal{V}, \psi, \eta)$ are two trivializations of X. We say that $(\mathcal{U}, \phi, \gamma)$ and $(\mathcal{V}, \psi, \eta)$ are *equivalent* if there exists some common refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$ such that the maps

$$\mathrm{id} : X \to X : x \mapsto x, \qquad \iota_i : W_i \to S : u \mapsto u,$$

define an S-bundle isomorphism $(\mathcal{W}, \mathrm{id}, \iota)$ from X with trivialization $(\mathcal{U}, \phi, \gamma)$ onto X with trivialization $(\mathcal{V}, \psi, \eta)$.

Remark 2.9. We won't need to use this, but it's easy to see that Definition 2.8 defines an equivalence relation on the set of trivializations of some bundle X.

Example 2.10. Suppose X is a principal S-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$. Then $(\mathcal{U}, \phi, \gamma)$ and $(\mathcal{V}, \phi_{\mathcal{V}}, \gamma_{\mathcal{V}})$ are equivalent.

We can now write down a better definition of principal S-bundle.

Definition 2.11. Suppose S is an abelian locally compact Hausdorff group bundle and X is a locally compact Hausdorff bundle over $S^{(0)}$. Suppose $\mathcal{A}$ is a set of triplets of the form $(\mathcal{U}, \phi, \gamma)$ such that X is a principal S-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$ for all $(\mathcal{U}, \phi, \gamma) \in \mathcal{A}$. Then $\mathcal{A}$ is called *pairwise equivalent* if any two trivializations in $\mathcal{A}$ are equivalent. A pairwise equivalent collection $\mathcal{A}$ is called *maximal* if it is not properly contained in any other pairwise equivalent collection. A locally compact Hausdorff bundle over $S^{(0)}$ equipped with a maximal pairwise equivalent collection of trivializations is called a *principal S-bundle*.

Remark 2.12. Every principal S-bundle X with fixed trivialization $(\mathcal{U}, \phi, \gamma)$ can be turned into a principal S-bundle by using Zorn's Lemma to find the maximal pairwise equivalent collection $\mathcal{A}$ containing $(\mathcal{U}, \phi, \gamma)$. It follows from Example 2.10 that any refinement of a trivialization in $\mathcal{A}$ is in $\mathcal{A}$.

Definition 2.13. Given two principal S-bundles X and Y we say X is *isomorphic* to Y if there is an isomorphism between the two bundles with respect to some pair of trivializations.

Remark 2.14. It is clear from Definition 2.11 that if two principal S-bundles X and Y are isomorphic with respect to some given pair of trivializations then they are isomorphic with respect to all trivializations. Furthermore, since any two trivializations in the maximal collections associated to X and Y are isomorphic with respect to the identity map, the isomorphism between X and Y is basically the same for all trivializations.

Next, we would like to mimic the group case and characterize the set of all principal S-bundles by classes in some cohomology group. We will be using sheaf cohomology as defined and developed in [RW98, Section 4.1].

Proposition 2.15. *Let S be an abelian locally compact Hausdorff group bundle and for U open in $S^{(0)}$ let $\mathcal{S}(U) = \Gamma(U, S)$ be the set of continuous sections from U into S. Then $\mathcal{S}$ is a sheaf and as such gives rise to a sheaf cohomology $H^n(S^{(0)}; \mathcal{S})$ which we shall denote by $H^n(S)$.*

Proof. First we show that $\mathcal{S}$ is a presheaf. Observe that $\Gamma(U, S)$ is non-empty for all U because the inclusion of U into S is always a section for p. It follows from the continuity of the operations on S that the pointwise multiplication of two continuous sections, or the inverse of a continuous section, is still a continuous section. It is easy to see that the group axioms hold with respect to these operations, with the inclusion of U into S acting as the identity.

Next, observe that $\Gamma(\emptyset, S) = \{0\}$ since there is only one "function" with the empty domain. Finally, given open subsets U and V of $S^{(0)}$ such that $U \subset V$ we can define $\rho_{V,U} : \Gamma(V, S) \to \Gamma(U, S)$ to be given by restriction to U. It is clear that $\rho_{U,U} = \mathrm{id}$ and that for $U \subset V \subset W$ we have

$$\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U}.$$

Thus $\mathcal{S}$ defines a presheaf on $S^{(0)}$.

Now suppose we have an open set $U \subset S^{(0)}$ and a decomposition $U = \bigcup_{i \in I} U_i$ of U into open sets U_i. Furthermore, suppose we have $\gamma_i \in \Gamma(U_i, S)$ for all $i \in I$ and for all $i, j \in I$

$$\rho_{U_i, U_{ij}}(\gamma_i) = \rho_{U_j, U_{ij}}(\gamma_j).$$

Tracing through the definitions we see that each γ_i is a continuous section on U_i such that the γ_i agree on overlaps. Therefore, we can define a continuous section γ on U in a piecewise fashion so that $\rho_{U,U_i}(\gamma) = \gamma_i$. Furthermore, it is clear that γ is uniquely determined by the γ_i.

Thus $\mathcal{S}$ is a sheaf of groups on $S^{(0)}$. Furthermore, because S is an abelian group bundle it's easy to see that each $\Gamma(U, S)$ is an abelian group and that $\mathcal{S}$ is an abelian sheaf. As such it has an associated cohomology described in [RW98, Section 4.1] which we use to define $H^n(S^{(0)}; \mathcal{S}) = H^n(S)$. □

At this point we can build the desired correspondence between principal S-bundles and elements of $H^1(S)$. The proof uses the details of the construction of $H^1(S)$. These details can be found in [RW98, Section 4.1] and anyone unfamiliar with sheaf cohomology should at least look through this section before working through the following proof.

Theorem 2.16. *Suppose S is an abelian locally compact Hausdorff group bundle. There is a one-to-one correspondence between the isomorphism classes of principal S-bundles and elements of the sheaf cohomology group $H^1(S)$. Given a principal bundle X with trivialization $(\mathcal{U}, \phi, \gamma)$, the cohomology class in $H^1(S)$ associated to X is realized by the cocycle γ.*

Proof. First, suppose X is a principal S-bundle and pick a trivialization $(\mathcal{U}, \phi, \gamma)$. Let $\mathcal{S}$ be the sheaf defined in Proposition 2.15. We will use the sheaf cohomology notation and definitions from [RW98, Section 4.1]. Observe that $\gamma_{ij} \in \mathcal{S}(U_{ij}) = \Gamma(U_{ij}, S)$ by definition. This implies that $\gamma = \{\gamma_{ij}\}$ forms a chain in $C^1(\mathcal{U}, \mathcal{S})$. We need to show that this chain is a cocycle; we need to show that $d(\gamma)$ is the trivial section. This amounts to proving that

$$\gamma_{ij}(u)\gamma_{jk}(u) = \gamma_{ik}(u)$$

for all $u \in U_{ijk}$. Well, given $u \in U_{ijk}$ we have

$$\begin{aligned}
\gamma_{ij}(u)\gamma_{jk}(u) &= \gamma_{ij}(u)\gamma_{jk}(u)u = \phi_i \circ \phi_j^{-1}(\phi_j \circ \phi_k^{-1}(u)) \\
&= \phi_i \circ \phi_k^{-1}(u) = \gamma_{ik}(u)u \\
&= \gamma_{ik}(u).
\end{aligned}$$

Thus γ is a cocycle and as such we can use γ to define a class $[\gamma] \in H^1(S^{(0)}; \mathcal{S})$.

We need to see that if we take a different trivialization for X then we get the same cohomology class. We start by showing that if we take a refinement of $\mathcal{U}$ then we don't change $[\gamma]$. Suppose $\mathcal{V}$ is a refinement of $\mathcal{U}$ with refining map r. Then the trivialization of X associated to $\mathcal{V}$ is, according to Proposition 2.5, given by $(\mathcal{V}, \phi_{\mathcal{V}}, \gamma_{\mathcal{V}})$ where, in particular, $(\gamma_{\mathcal{V}})_{ij} = \gamma_{r(i)r(j)}|_{V_{ij}}$. It follows, by definition, that $\gamma_{\mathcal{V}} = r_*(\gamma)$ where $r_* : H^n(\mathcal{U}, \mathcal{S}) \to H^n(\mathcal{V}, \mathcal{S})$ is the homomorphism defined by restriction. Since

$H^n(S^{(0)};\mathcal{S})$ is a direct limit of the $H^n(\mathcal{U};\mathcal{S})$ with respect to the restriction maps r_* it is clear that $[\gamma] = [\gamma_{\mathcal{V}}]$ so that we can safely pass to refinements without changing the associated cohomology class.

Now suppose that Y is another bundle isomorphic to X. Suppose $(\mathcal{U}, \phi, \gamma)$ is a trivialization for X and $(\mathcal{V}, \psi, \eta)$ a trivialization for Y. In particular, Y could be X with an equivalent trivialization. The goal is to show that the cohomology classes associated to X and Y with these trivializations are equal. Let $(\mathcal{W}, \Omega, \beta)$ be an isomorphism from X to Y. By passing to a common refinement and using the previous paragraph we can assume, without loss of generality, that $\mathcal{U} = \mathcal{V} = \mathcal{W}$. Then, for all $u \in U_{ij}$,

$$\begin{aligned}
\eta_{ij}(u)\beta_j(u) &= \eta_{ij}(u)\beta_j(u)u \\
&= \psi_i \circ \psi_j^{-1}(\psi_j \circ \Omega \circ \phi_j^{-1}(u)) \\
&= \psi_i \circ \Omega \circ \phi_i^{-1}(\phi_i \circ \phi_j^{-1}(u)) \\
&= \beta_i(u)\gamma_{ij}(u)u \\
&= \beta_i(u)\gamma_{ij}(u).
\end{aligned}$$

Hence $\gamma^{-1}\eta$ is a boundary and therefore $[\gamma] = [\eta]$ in $H^1(S^{(0)};\mathcal{S})$. This proves a number of things. First, it shows that no matter which trivialization for X we choose we get the same cohomology class in $H^1(S^{(0)},\mathcal{S})$. Therefore we can define a map $X \mapsto [X]$ from the set of principal S-bundles to $H^1(S^{(0)};\mathcal{S})$ by $[X] = [\gamma]$ where γ is the cocycle defined by any set of transition maps for X. Furthermore, since we were working with an arbitrary isomorphic bundle Y, this also shows that the map $X \mapsto [X]$ is a well defined function from the set of isomorphism classes of principal S-bundles into $H^1(S)$.

Now we are going to construct an inverse map. Suppose $c \in H^1(S^{(0)};\mathcal{S})$ is realized by $\gamma \in Z^1(\mathcal{U},\mathcal{S})$ for some open cover $\mathcal{U}$. Let $C = \coprod_i p^{-1}(U_i)$ be the disjoint union of the $p^{-1}(U_i)$ and denote elements of C by (s,i) where $s \in p^{-1}(U_i)$. Observe that we can define $\tilde{p} : C \to S^{(0)}$ by by $\tilde{p}(s,i) = p(s)$, and that this map will be continuous since

$$\tilde{p}^{-1}(O) = \bigcup_i p^{-1}(U_i \cap O) \times \{i\}$$

for any open set $O \subset S^{(0)}$. From here on we will often denote the clopen subset $p^{-1}(U_i) \times \{i\}$ of C by $p^{-1}(U_i)$. Since $p^{-1}(U_i) \times \{i\}$ and $p^{-1}(U_i)$ are clearly homeomorphic there should be relatively little confusion. Define a relation on C by $(s,i) \equiv (t,j)$ if and only if $p(s) = p(t) = u$ and $s = \gamma_{ij}(u)t$. We need to show that this is an equivalence relation. Since γ is a cocycle we know that

$$\gamma_{ij}\gamma_{jk} = \gamma_{ik} \tag{2.1}$$

for all i, j, k. In particular (2.1) implies $\gamma_{ii}\gamma_{ii} = \gamma_{ii}$ so that for all $u \in U_i$ we have $\gamma_{ii}(u) = \gamma_{ii}(u)^{-1}\gamma_{ii}(u)$ and $\gamma_{ii}(u) \in S^{(0)}$. However, because γ_{ii} is a section, this implies that $\gamma_{ii}(u) = u$ for all $u \in U_i$ and that γ_{ii} is just the inclusion of U_i into S. It follows that $(s, i) \equiv (s, i)$ for all $(s, i) \in C$. Since γ_{ii} is inclusion we have $\gamma_{ij}(u)\gamma_{ji}(u) = \gamma_{ii}(u) = u$. This implies $\gamma_{ij}(u)^{-1} = \gamma_{ji}(u)$ for all $u \in U_{ij}$ and this is enough to show that $\equiv$ is symmetric. Finally, it is easy enough to use the cocycle identity (2.1) to show that $\equiv$ is transitive. Let X_γ be the quotient of C by $\equiv$ with equivalence classes denoted by $[s, i]$ for $(s, i) \in C$ and associated quotient map Q. Since $\tilde{p}$ is constant on $[s, i]$, we can factor $\tilde{p}$ through Q to obtain a continuous surjection $q : X_\gamma \to S^{(0)}$ such that $q([s, i]) = p(s)$.

Now, it is clear that $Q(p^{-1}(U_i)) \subset q^{-1}(U_i)$. If $[s, j] \in q^{-1}(U_i)$ then $u = p(s) \in U_{ij}$ and $(s, j) \equiv (\gamma_{ji}(u)s, i)$. However, $(\gamma_{ji}(u)s, i) \in p^{-1}(U_i)$ and therefore $Q(p^{-1}(U_i)) = q^{-1}(U_i)$. Next, observe that if $(s, i) \equiv (t, i)$ then $s = \gamma_{ii}(p(s))t = t$. Since the equivalence relation is trivial on $p^{-1}(U_i)$ the restriction $Q|_{p^{-1}(U_i)} : p^{-1}(U_i) \to q^{-1}(U_i)$ is a continuous bijection, which we will denote Q_i. Suppose O is open in $p^{-1}(U_i)$. In order to show that $Q_i(O)$ is open we must show $Q^{-1}(Q_i(O)) = Q^{-1}(Q(O))$ is open. Suppose $(t_l, j_l) \to (t, j)$ in C and $(t, j) \in Q^{-1}(Q(O))$. Since C is a disjoint union, $j_l = j$ eventually. So, disregarding some initial segment, we can assume $j_l = j$ for all j. Now $(t, j) \in Q^{-1}(Q(O))$ so there exits $(s, i) \in O$ such that $(t, j) \equiv (s, i)$. In particular this implies that $u = p(s) = p(t) \in U_{ij}$. Let $u_l = p(t_l)$ and observe that $u_l \to u$ so that eventually $u_l \in U_{ij}$. As before we can assume without loss of generality that this is always true. Since $u_l \in U_{ij}$ we can define $s_l = \gamma_{ij}(u_l)t_l$. Because $\gamma_{ij}(u_l) \to \gamma_{ij}(u)$ and $t_l \to t$ we have $s_l \to s$ so that eventually $s_l \in O$. Then $(s_l, i) \equiv (t_l, j)$ implies that, eventually, $(t_l, j) \in Q^{-1}(Q(O))$ so that $Q^{-1}(Q(O))$ is open. Hence Q_i is a homeomorphism from $p^{-1}(U_i)$, which we now view as a subset of S, onto $q^{-1}(U_i)$.

First, note that because X_γ is locally homeomorphic to S it is straightforward to show that X_γ is locally compact Hausdorff, and that we can view X_γ as a bundle over $S^{(0)}$ with bundle map q. We will show that X_γ is a principal S-bundle. Let $\phi_i = Q_i^{-1}$ and observe that since $q \circ Q = p$ we have $q = p \circ \phi_i$. Furthermore, given $s \in p^{-1}(U_{ij})$

$$\begin{aligned} \phi_i \circ \phi_j^{-1}(s) &= \phi_i([s, j]) \\ &= Q_i^{-1}([\gamma_{ij}(p(s))s, i]) = \gamma_{ij}(p(s))s \end{aligned}$$

where the second equality holds because, by definition, $(s, j) \equiv (\gamma_{ij}(p(s))s, i)$. Therefore, the ϕ_i define local trivializations of X_γ and the transition maps are the γ_{ij}. Hence X_γ is a principal S-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$. Furthermore, it is clear from our construction that the cohomology class associated to X_γ is $[X_\gamma] = [\gamma] = c$.

Next, we must see that our map is well defined in the sense that if we choose two different realizations of c we end up with isomorphic principal bundles. Let $\eta = \{\eta_{ij}\}$

be some other cocycle which implements c on an open cover $\mathcal{V}$. Since $[\eta] = [\gamma] = c$ we can pass to some common refinement of $\mathcal{U}$ and $\mathcal{V}$, say $\mathcal{W}$ with refining maps r and ρ respectively, and find continuous sections $\beta_i \in \Gamma(W_i, S)$ such that

$$\eta_{\rho(i)\rho(j)}\beta_j = \beta_i\gamma_{r(i)r(j)}.$$

We define $\Omega : X_\gamma \to X_\eta$ locally by $\Omega([s, r(i)]) = [\beta_i(p(s))s, \rho(i)]$. This is well defined because if $(s, r(i)) \equiv (t, r(j))$ then $p(s) = p(t) = u$ and

$$\beta_i(u)s = \beta_i(u)\gamma_{r(i)r(j)}(u)t = \eta_{\rho(i)\rho(j)}\beta_j(u)t$$

so that $(\beta_i(u)s, \rho(i)) \equiv (\beta_j(u)t, \rho(j))$. Since each β_i is continuous, Ω is locally continuous and therefore continuous. It's easy to see that we can construct a continuous inverse for Ω by using β_i^{-1} and that Ω is a homeomorphism. Furthermore, it is clear that Ω preserves fibres. Finally, given $s \in p^{-1}(W_i)$ we have

$$\psi_{\rho(i)} \circ \Omega \circ \phi_{r(i)}^{-1}(s) = \psi_{\rho(i)} \circ \Omega([s, r(i)]) = \psi_{\rho(i)}([\beta_i(p(s))s, \rho(i)]) = \beta_i(p(s))s.$$

It follows that $(\mathcal{W}, \Omega, \beta)$ is an isomorphism from X_γ to X_η. Thus we have constructed a well defined map $[\gamma] \mapsto X_\gamma$ from $H^1(S)$ into the set of isomorphism classes of principal S-bundles. Furthermore, it is clear that this map is a right inverse for $X \mapsto [X]$.

The last thing we must show is that this is a left inverse. Suppose X is a principal S-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$ and X_γ is constructed as above. Define $\Omega : X \to X_\gamma$ by $\Omega(x) = [\phi_i(x), i]$ when $q(x) \in U_i$. Suppose we have $q(x) \in U_j$ as well. Then $\phi_i^{-1}(\phi_j(s)) = \gamma_{ij}(p(s))s$ for all $s \in p^{-1}(U_{ij})$ so that for all $x \in q^{-1}(U_{ij})$ we get

$$\phi_i(x) = \gamma_{ij}(q(x))\phi_j(x).$$

However, this shows that $(\phi_i(x), i) \equiv (\phi_j(x), j)$ and that Ω is well defined. Furthermore, it is easy to see by construction that Ω preserves the fibres. Next, if $x_l \to x$ and $q(x) \in U_i$ then, eventually, $q(x_l) \in U_i$ and $\Omega(x_l) = [\phi_i(x_l), i]$. Now, $\phi_i(x_l) \to \phi_i(x)$ and it follows then that $\Omega(x_l) \to \Omega(x)$ so that Ω is continuous. Furthermore, it is easy to see that we can construct a continuous inverse via the map $[s, i] \mapsto \phi_i^{-1}(s)$ so that Ω is a homeomorphism. Now let ψ_i be the local trivializations for X_γ and observe that for $s \in p^{-1}(U_i)$

$$\psi_i \circ \Omega \circ \phi_i^{-1}(s) = \psi_i([s, i]) = s.$$

Therefore letting ι_i be inclusion turns $(\mathcal{U}, \Omega, \iota)$ into an isomorphism of principal S-bundles. It follows that X and X_γ have the same isomorphism class and that we have constructed a bijection between these classes and $H^1(S)$. $\square$

Remark 2.17. Given an abelian locally compact Hausdorff group bundle S we can view S as a principal S-bundle with respect to any cover $\mathcal{U}$ by letting ϕ be the identity map and γ_{ij} be the inclusion of U_{ij} into S. Since the trivial class in $H^1(S)$ is realized on any cover $\mathcal{U}$ by the inclusion maps of U_{ij} into S we see that the identity is the cohomological invariant associated to the class of trivial S-bundles.

We continue our exploration of principal S-bundles by showing that they are equivalent to a certain class of principal S-spaces.

Proposition 2.18. *Suppose X is a principal S-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$. Define the source map on X to be its bundle map. Then, for $s \in S$, $x \in X$ such that $p(s) = q(x) \in U_i$,*

$$s \cdot x = \phi_i^{-1}(s\phi_i(x))$$

defines a continuous action of S on X. Furthermore the following hold:

(a) *The action of S on X is principal.*

(b) *For all i the map ϕ_i is equivariant with respect to this action and the action of S on itself by left multiplication.*

(c) *The action of S on X is orbit transitive.*

Proof. First, we need to make sure the action is well defined on overlaps. Suppose $u = q(x) = p(s)$ and $u \in U_{ij}$. Then, using the fact that S is abelian,

$$\begin{aligned}\phi_i^{-1}(s\phi_i(x)) &= \phi_j^{-1} \circ \phi_j \circ \phi_i^{-1}(s\phi_i(x)) = \phi_j^{-1}(\gamma_{ji}(u)s\phi_i(x)) \\ &= \phi_j^{-1}(s\phi_j(\phi_i^{-1}(\phi_i(x)))) = \phi_j^{-1}(s\phi_j(x)).\end{aligned}$$

Hence the action is well defined. Now suppose $s, t \in S$ and $x \in X$ such that $p(s) = p(t) = q(x) = u \in U_i$. Then

$$\begin{aligned}s \cdot (t \cdot x) &= s \cdot (\phi_i^{-1}(t\phi_i(x))) = \phi_i^{-1}(s\phi_i(\phi_i^{-1}(t\phi_i(x)))) \\ &= \phi_i^{-1}(st\phi_i(x)) = st \cdot x.\end{aligned}$$

It is also easy to see that $q(x) \cdot x = x$ for all $x \in X$. Next, suppose $s_l \to s$ and $x_l \to x$ such that $p(s_l) = q(x_l) = u_l$ for all l and $p(s) = q(x) = u \in U_i$. Eventually $u_l \in U_i$ and over U_i the action is clearly continuous. Therefore the action of S on X is a continuous groupoid action.

Part **(a)**: Suppose $s \cdot x = x$ for $s \in S$ and $x \in X$. Then for some i we have $\phi_i^{-1}(s\phi_i(x)) = x$ so that $s\phi_i(x) = \phi_i(x)$. It follows that $s \in S^{(0)}$ and that the action is free. Now suppose x_l and s_l are nets in X and S respectively so that $x_l \to x$ and $s_l \cdot x_l \to y$. We can pass to a subnet and assume that $p(s) = q(x) = q(y) = u \in U_i$ and $p(s_l) = q(x_l) = u_l \in U_i$ for all l. In this case $s_l\phi_i(x_l) \to \phi_i(y)$ and, combining

this with the fact that $\phi_i(x_l) \to \phi_i(x)$, we have $s_l \to \phi_i(y)\phi_i(x)^{-1}$. It follows that the action of S on X is proper, and therefore principal.

Part **(b)**: Suppose $s \in S$ and $x \in X$ such that $p(s) = q(x) = u \in U_i$. Then

$$\phi_i(s \cdot x) = \phi_i(\phi_i^{-1}(s\phi_i(x))) = s\phi_i(x)$$

so that ϕ is equivariant with respect to the action of S on X and the action of S on itself by left multiplication.

Part **(c)**: Suppose $x, y \in X$ such that $q(x) = q(y) = u$. We need to find $s \in S$ such that $s \cdot x = y$. Choose U_i so that $u \in U_i$ and let $s = \phi_i(y)\phi_i(x)^{-1}$. Then we are done since

$$s \cdot x = \phi_i^{-1}(\phi_i(y)\phi_i(x)^{-1}\phi_i(x)) = y. \qquad \square$$

This next proposition shows that we can view principal S-bundles as particularly nice S-spaces. From now on we will think of principal S-bundles in this manner.

Theorem 2.19. *Suppose S is an abelian locally compact Hausdorff group bundle and X is a locally compact Hausdorff space. Then X is a principal S-bundle if and only if X is a principal, orbit transitive, S-space such that the range map on X has local sections.*

Proof. If X is a principal X-bundle then let $(\mathcal{U}, \phi, \gamma)$ be a trivialization of X and let S act on X as in Proposition 2.18. All we need to show is that q has local sections. On U_i define $\sigma_i : U_i \to X$ by $\sigma_i(u) = \phi_i^{-1}(u)$. It is easy to see that σ_i is a continuous section of q on U_i and we are done.

Next, suppose S acts on X as in the statement of the theorem and that $\mathcal{U}$ is an open cover of $S^{(0)}$ such that there are local sections $\sigma_i : U_i \to X$ of q. We define $\psi_i : p^{-1}(U_i) \to q^{-1}(U_i)$ by $\psi_i(s) = s \cdot \sigma_i(p(s))$. Since everything in sight is continuous, it is clear that ψ_i is continuous. Furthermore, using Proposition 1.59,

$$q(\psi_i(s)) = q(s \cdot \sigma_i(p(s))) = q(\sigma_i(p(s))) = p(s).$$

Now, if $\psi_i(s) = \psi_i(t)$ then we have $p(s) = p(t)$ and, after multiplying by t^{-1},

$$t^{-1}s \cdot \sigma_i(p(s)) = \sigma_i(p(s)).$$

Since the action is free this implies $t^{-1}s \in S^{(0)}$ and $t = s$ so ϕ_i is injective. Next, if $y \in q^{-1}(U_i)$ then $q(y) = q(\sigma_i(q(y)))$ so that, by orbit transitivity, there exists $s \in S$ such that $y = s \cdot \sigma_i(q(y))$. It is clear that $\psi_i(s) = y$ and that ψ_i is surjective. Now suppose $\psi_i(s_l) \to \psi_i(s)$. Pass to a subnet. We will show that there is a sub-subnet such that $s_l \to s$. By definition we have $s_l \cdot \sigma_i(p(s_l)) \to s \cdot \sigma_i(p(s))$. Furthermore $q(s_l \cdot \sigma_i(p(s_l))) = p(s_l)$ for all l and $q(s \cdot \sigma_i(p(s))) = p(s)$. Since q is continuous we have $p(s_l) \to p(s)$ and therefore $\sigma_i(p(s_l)) \to \sigma_i(p(s))$. Since the action of S is proper this

implies that we can pass to a subnet, relabel, and find t such that $s_l \to t$. However, using the continuity of the action, this implies $s_l \cdot \sigma_i(p(s_l)) \to t \cdot \sigma_i(p(s))$. Using the fact that X is Hausdorff and the action is free we have $s = t$ and we are done. Therefore ψ_i is a homeomorphism and we define the trivializing maps to be $\phi_i = \psi_i^{-1}$.

Next, we need to compute the transition functions. Suppose $s \in p^{-1}(U_{ij})$. Then

$$\begin{aligned}\phi_i \circ \phi_j^{-1}(s) &= \psi_i^{-1} \circ \psi_j(s) = \psi_i^{-1}(s \cdot \sigma_j(p(s))) \\ &= \psi_i^{-1}(\gamma_{ij}(p(s))s \cdot \sigma_i(p(s))) = \gamma_{ij}(p(s))s\end{aligned}$$

where $\gamma_{ij}(u)$ is the unique element of S such that

$$\gamma_{ij}(u) \cdot \sigma_i(u) = \sigma_j(u).$$

We know $\gamma_{ij}(u)$ is guaranteed to exist because the action is orbit transitive and that $\gamma_{ij}(u)$ is unique because the action is free. Since γ_{ij} is clearly a section of S on U_{ij} and since it satisfies the right algebraic properties, all that is left is to show it is continuous. However, if $u_l \to u$ in U_{ij} then $\sigma_i(u_l) \to \sigma_i(u)$ and

$$\sigma_j(u_l) = \gamma_{ij}(u_l) \cdot \sigma_i(u_l) \to \sigma_j(u) = \gamma_{ij}(u) \cdot \sigma_i(u).$$

Using properness as we did before, it is easy to show that $\gamma_{ij}(u_l) \to \gamma_{ij}(u)$. It follows that X is a principal S-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$. $\square$

Remark 2.20. As in the classical case, trivial principal S-bundles are exactly principal, orbit transitive S-spaces whose range maps have a *global* section.

This next proposition is nice because it frees our idea of principal bundle isomorphism from the hassle of having to keep track of local trivializations. It is also mildly remarkable that Ω is not required to be a homeomorphism, or even a bijection.

Proposition 2.21. *Suppose X and Y are principal S-bundles. Then X and Y are isomorphic if and only if there exists a continuous map $\Omega : X \to Y$ which is S-equivariant with respect to the actions of S on X and Y.*

Proof. Let X and Y be as above with bundle maps q and r, and trivializations $(\mathcal{U}, \phi, \gamma)$ and $(\mathcal{V}, \psi, \eta)$ respectively. Suppose $(\mathcal{W}, \Omega, \beta)$ is a principal bundle isomorphism from X to Y. Definition 2.7 requires that $q = r \circ \Omega$ so that Ω respects the range maps on X and Y. It is obvious that Ω is continuous. Now if $s \in S$ and $x \in X$ such that $p(s) = q(x) \in U_i$ then

$$\begin{aligned}\Omega(s \cdot x) &= \psi_i^{-1} \circ \psi_i \circ \Omega \circ \phi_i^{-1}(s\phi_i(x)) \\ &= \psi_i^{-1}(\beta_i(u)s\phi_i(x)) \\ &= \psi_i^{-1}(s\psi_i \circ \Omega \circ \phi_i^{-1}(\phi_i(x)))\end{aligned}$$

$$= \psi_i^{-1}(s\psi_i(\Omega(x))) = s \cdot \Omega(x).$$

Hence Ω is a continuous equivariant map.

Now suppose $\Omega : X \to Y$ is a continuous equivariant map. Since the range maps of X and Y are precisely their bundle maps, it is part of Definition 1.57 that Ω preserves the fibres. Let $(\mathcal{U}, \phi, \gamma)$ and $(\mathcal{V}, \phi, \eta)$ be trivializations for X and Y respectively. By passing to a common refinement we may assume without loss of generality that $\mathcal{U} = \mathcal{V}$. Given U_i let $\Omega_i = \phi_i \circ \Omega \circ \phi_i^{-1}$. Since each of its component maps preserves fibres Ω_i does as well, and therefore $\Omega_i|_{S_u}$ maps S_u into S_u for $u \in U_i$. Suppose $s, t \in S_u$, then

$$\begin{aligned}
\Omega_i(st) &= \psi_i \circ \Omega \circ \phi_i^{-1}(s\phi_i(\phi_i^{-1}(t))) = \psi_i \circ \Omega(s \cdot \phi_i^{-1}(t)) \\
&= \psi_i(s \cdot \Omega(\phi_i^{-1}(t)) = \psi_i(\psi_i^{-1}(s \cdot \psi_i \circ \Omega \circ \phi_i^{-1}(t))) \\
&= s\Omega_i(t).
\end{aligned}$$

Now, the following general nonsense implies any map h from a group H into itself such that h is equivariant with respect to the action of H on itself by left multiplication is given by left multiplication. That is, given map $h : H \to H$ such that $h(st) = sh(t)$ for all $s, t \in H$, we have

$$h(s) = h(es) = h(e)s.$$

Hence, h is actually just left multiplication by $h(e)$. Applying this to the current situation we find that $\Omega_i|_{S_u}$ is given on S_u by left multiplication by $\Omega_i(u)$. Define β_i on U_i by $\beta_i(u) = \Omega_i(u)$. The function β_i is a section of S on U_i which is continuous because Ω_i is. Since Ω_i is defined by left multiplication of the continuous section β_i it follows immediately that Ω_i has a continuous inverse given by left multiplication by β_i^{-1}. Thus Ω_i is a homeomorphism. But then $\Omega|_{q^{-1}(U_i)} = \psi_i^{-1} \circ \Omega_i \circ \phi_i$ is a homeomorphism for all i. It is straightforward to show that this implies that Ω is a homeomorphism. Furthermore, we know that for $s \in q^{-1}(U_i)$

$$\psi_i \circ \Omega \circ \phi_i^{-1}(s) = \Omega_i(s) = \beta_i(p(s))s.$$

Hence $(\mathcal{U}, \Omega, \beta)$ is a principal bundle isomorphism of X onto Y. □

It is philosophically important to see that the theory of principal S-bundles is an extension of the classical theory of principal group bundles. Since we will not use the classical theory of principal group bundles, we will not reproduce those definitions here and will instead refer the reader to [RW98, Section 4.2]. As with Theorem 2.16, the following will draw heavily from [RW98].

Proposition 2.22. *Suppose H is an abelian locally compact Hausdorff group and X and Y are locally compact Hausdorff spaces. Let $S = Y \times H$ be the trivial group bundle. Then X is a principal H-bundle over Y if and only if X is a principal*

S-bundle.

Proof. According to [RW98] a principal H-bundle over Y is just a fibre bundle X with fibres H and structure group H where H acts on itself by left multiplication. In particular, if $q : X \to Y$ is a principal H-bundle then there exists an open cover $\mathcal{U}$, homeomorphisms $\phi_i : q^{-1}(U_i) \to U_i \times H$ such that $\phi_i(x)$ has the form $(q(x), s)$ for some $s \in H$, and continuous maps $\gamma_{ij} : U_{ij} \to H$ such that

$$\phi_i \circ \phi_j^{-1}(x, s) = (x, \gamma_{ij}(x)s) \tag{2.2}$$

for all $x \in U_{ij}$ and $s \in H$. However, if we view $S = Y \times H$ as a group bundle over Y with bundle map p then $q^{-1}(U_i) = U_i \times H$ and ϕ_i is nothing more than a bundle homeomorphism from $q^{-1}(U_i)$ onto $p^{-1}(U_i)$. It is then clear from (2.2) that the ϕ_i form trivializing maps and the maps $\tilde{\gamma}_{ij}$ defined in the natural way by $\tilde{\gamma}_{ij}(y) = (y, \gamma_{ij}(y))$ form transition maps so that X is a principal S-bundle with trivialization $(\mathcal{U}, \phi, \tilde{\gamma})$. This exact same argument holds in reverse, and it's straightforward to show that if X is a principal S-bundle then the trivialization $(\mathcal{U}, \phi, \gamma)$ makes X into a principal H-bundle. □

In particular, it's nice to observe that the sheaf cohomology theory associated to principal S-bundles is just an extension of the group sheaf cohomology theory associated to principal H-bundles.

Proposition 2.23. *Suppose H is an abelian locally compact Hausdorff group, Y a locally compact Hausdorff space, and S is the trivial bundle $Y \times H$. Then $H^n(S) \cong H^n(Y; H)$ and under this identification X generates the same cohomology class when viewed as either a principal S-bundle or a principal H-bundle.*

Proof. By Proposition 2.15 we know $H^n(S)$ is the sheaf cohomology generated by the sheaf $\mathcal{S}$ where $S(U) = \Gamma(U, S)$ is the set of continuous sections on U for a given open set $U \in Y$. The sheaf cohomology of Y with coefficients H, denoted $H^n(Y; H)$, is the sheaf cohomology associated to the sheaf $\mathcal{T}$ where $\mathcal{T}(U) = C(U, H)$ is the set of continuous functions from U into H. Given $f \in C(U, H)$ we can define $\tilde{f} \in \Gamma(U, S)$ by $\tilde{f}(y) = (y, f(y))$. It's easy to see that this map is an isomorphism of $\mathcal{T}(U)$ onto $\mathcal{S}(U)$ and extends to a sheaf isomorphism of $\mathcal{T}$ onto $\mathcal{S}$. This allows us to identify the cohomology groups $H^n(S) = H^n(Y, \mathcal{S})$ and $H^n(Y; H) = H^n(Y; \mathcal{T})$ via $[\gamma] \mapsto [\tilde{\gamma}]$.

Now suppose $q : X \to Y$ is a topological bundle. Proposition 2.23 tells us that X is a principal H-bundle if and only if it's a principal Y-bundle. Furthermore, it follows from the proof of Proposition 2.23 that if γ_{ij} are the transition maps for X as a principal H-bundle then $\tilde{\gamma}_{ij}$ are the transition maps for X as a principal S-bundle. We showed in Theorem 2.16 that the cohomology invariant associated to X as a principal S-bundle is $[\tilde{\gamma}_{ij}]$. It is shown in [RW98, Proposition 4.53] that the

cohomology invariant associated to X as a principal H-bundle is $[\gamma_{ij}]$. Since these two classes are precisely the ones identified under our isomorphism, we are done. $\square$

2.1.1 Locally σ-trivial Spaces

Principal "group bundle bundle" theory is a natural extension of classical principal group bundle theory. The real question is if there are principal S-bundles which are not generated by principal H-bundles. Fortunately, principal S-bundles are also an extension of the notion of σ-trivial spaces as defined in [RW88] and there are nontrivial examples given there. We will include these examples as part of this section for completeness. However, first we have to define σ-trivial spaces.

Definition 2.24. Suppose the abelian locally compact Hausdorff group H acts on the locally compact Hausdorff space X and the stabilizers vary continuously in H. We shall say that X is a locally σ-trivial space if X/H is Hausdorff and if every $x \in X$ has a G-invariant neighborhood U such that there exists $\phi : U \to (U/H \times H)/\cong$ where

$$(H \cdot x, s) \cong (H \cdot y, t) \text{ if and only if } H \cdot x = H \cdot y \text{ and } st^{-1} \in H_x.$$

Furthermore we require that

(a) If $x \in U$ then $\phi(x) = [H \cdot x, s]$ for some $s \in H$ and,

(b) If $x \in U$ and $s \in H$ and $\phi(x) = [H \cdot x, t]$ then

$$\phi(s \cdot x) = [H \cdot x, st].$$

Our goal will be to construct a continuously varying abelian group bundle S associated to G and X. We start with a useful lemma.

Lemma 2.25. *Suppose H, X and $\cong$ are as in Definition 2.24. Then $\cong$ is an equivalence relation, the orbit space $S = (X/H \times H)/\cong$ is locally compact Hausdorff, and the quotient map from $X/H \times H$ onto S is open.*

Proof. First, given $s \in H$ and $x \in X$ suppose $t \in H_x$. Since H is abelian $t \cdot (s \cdot x) = st \cdot x = s \cdot x$. Hence $t \in H_{s \cdot x}$ and $H_x \subset H_{s \cdot x}$. We can be tricky and apply this argument to $s \cdot x$ and s^{-1} to see that the inclusion is actually an equality.

Now, we must show that $\cong$ is an equivalence relation. It is clear that $(H \cdot x, s) \cong (H \cdot x, s)$. Suppose $(H \cdot x, s) \cong (H \cdot y, t)$. Then, since $H \cdot x = H \cdot y$, by the previous paragraph, $H_x = H_y$ and since H_y is a group $st^{-1} \in H_y$ implies $ts^{-1} \in H_y$. Hence $(H \cdot y, t) \cong (H \cdot x, s)$. Finally, suppose $(H \cdot x, s) \cong (H \cdot y, t)$ and $(H \cdot y, t) \cong (H \cdot z, r)$.

Then, $H \cdot x = H \cdot y = H \cdot z$, and therefore $H_x = H_y = H_z$. By assumption st^{-1} and tr^{-1} are elements of H_x, but of course this implies $sr^{-1} \in H_x$ and we are done.

Let $Q : X/H \times H \to S$ be the quotient map and denote elements of S by $[H \cdot x, s]$. Suppose U is open in X/H and V is open in H. We must show that $O = Q^{-1}(Q(U \times V))$ is open. Notice that a generic element of O has the form $(H \cdot x, st)$ where $H \cdot x \in U$, $s \in V$, and $t \in H_x$. Suppose to the contrary that O is not open. This implies that the complement of O is not closed so that we can find a net such that $(H \cdot x_i, s_i) \notin O$ for all i, $(H \cdot x, st) \in O$, and such that $(H \cdot x_i, s_i) \to (H \cdot x, st) \in O$. In particular, this implies that $H \cdot x_i \to H \cdot x$. Since the quotient map from $X \to X/H$ is always open we can pass to a subnet, reindex, and possibly choose new representatives, so that $x_i \to x$. Since the stabilizers of H vary continuously with respect to the Fell topology this implies $H_{x_i} \to H_x$. However, $t \in H_x$ so, by Proposition 1.38, we can pass to another subnet, reindex, and find $t_i \in H_{x_i}$ such that $t_i \to t$. Since $s_i \to st$ it follows that $s_i t_i^{-1} \to s$ and since $s \in O$ this implies $s_i t_i \in O$ eventually. It follows that, eventually, $(H_{x_i}, s_i) \in O$, which is a contradiction. Therefore O, and hence Q, must be open.

Since Q is open and $X/H \times H$ is locally compact, it follows that S is locally compact. All that's left is to show that it is Hausdorff. Suppose $[H \cdot x_i, s_i]$ is a net in S which converges to both $[H \cdot x, s]$ and $[H \cdot y, t]$. Using the fact that Q is open we can pass to a subnet, twice actually, reindex, choose new representatives s_i, and find $t_i \in H_{x_i}$ such that

$$\begin{aligned} H \cdot x_i &\to H \cdot x & s_i &\to s \\ H \cdot x_i &\to H \cdot y & s_i t_i &\to t. \end{aligned}$$

Now, X/H is assumed to be Hausdorff so that $H \cdot x = H \cdot y$. We pass to yet another subnet, relabel, and choose new x_i so that $x_i \to x$. Since the stabilizers vary continuously this implies $H_{x_i} \to H_x$. Observe that

$$s_i^{-1}(s_i t_i) = t_i \to s^{-1} t$$

and that, via Proposition 1.38 because $t_i \in H_{x_i}$ for all i, we must have $s^{-1}t \in H_x$. This implies that $(H \cdot x, s) \cong (H \cdot x, t)$ and we are done. □

Now that we know the quotient space $(X/H \times H)/\cong$ is well behaved topologically we can prove a more interesting proposition.

Proposition 2.26. *Suppose the abelian locally compact Hausdorff group H acts on the locally compact Hausdorff space X. Furthermore, suppose the stabilizers vary continuously in H and that X/H is Hausdorff. Define $S_{(X,H)} := (X/H \times H)/\cong$, often denoted S, where $\cong$ is defined as in Definition 2.24. Then S is an abelian, continuously varying, locally compact Hausdorff group bundle whose unit space can*

be identified with X/H. The bundle map is given by $p([H \cdot x, s]) = H \cdot x$ and the operations are

$$[H \cdot x, s][H \cdot x, t] := [H \cdot x, st], \qquad [H \cdot x, s]^{-1} := [H \cdot x, s^{-1}].$$

The fibre $S_{H \cdot x}$ over $H \cdot x$ is (isomorphic to) H/H_x.

Proof. Define S as in the statement of the proposition and observe that it follows from Lemma 2.25 that S is locally compact Hausdorff. Furthermore, let $Q : X/H \times H \to S$ be the quotient map and recall that we showed that Q is open. Define

$$S^{(2)} = \{([H \cdot x, s], [H \cdot y, t]) \in S \times S : H \cdot x = H \cdot y\}$$

and observe that, since all representatives of the class $[H \cdot x, s]$ are of the form $(H \cdot x, st)$ where $t \in H_x$, $S^{(2)}$ is well defined. We would like to show that the operations given above are well defined. Suppose $(H \cdot x, s), (H \cdot x, t), (H \cdot y, s'), (H \cdot y, t') \in X/H \times H$ and that $(H \cdot x, s) \cong (H \cdot y, s')$ and $(H \cdot x, t) \cong (H \cdot y, t')$. Then $H \cdot x = H \cdot y$ and there exists $u, v \in H_x$ such that $s' = su$ and $t' = tv$. Since $s't' = (su)(tv) = st(uv)$ it follows that $(H \cdot x, st) \cong (H \cdot y, s't')$ and that the multiplication is well defined. The proof that the inverse operation is well defined is similar. It is easy to use the fact that H is a group to prove that S, with these operations, is a groupoid. Furthermore, the range and source maps on S are equal and are both given by

$$p([H \cdot x, s]) = [H \cdot x, e]$$

where $e \in H$ is the identity. Thus S is a group bundle and it is straightforward to see that $S^{(0)}$ can be identified with X/H via the map $[H \cdot x, e] \mapsto H \cdot x$. Furthermore, under this identification the bundle map p has the required form.

Next, we have to show that the operations are continuous. Suppose $[H \cdot x_i, s_i] \to [H \cdot x, s]$ and $[H \cdot x_i, t_i] \to [H \cdot x, t]$ in S. We can pass to a subnet, twice, relabel, and choose new representatives s_i and t_i such that $H \cdot x_i \to H_x$, $s_i \to s$ and $t_i \to t$. It follows immediately that

$$(H \cdot x_i, s_i t_i) \to (H \cdot x, st).$$

Showing that the inverse operation is continuous is a similar exercise. Suppose $H \cdot x_i \to H \cdot x$ in X/H and that $[H \cdot x, s] \in S$. Then $(H \cdot x_i, s) \to (H \cdot x, s)$ in $X/H \times H$. It follows immediately that the bundle map p is open so that S is a continuously varying locally compact Hausdorff group bundle.

Given $H \cdot x \in X/H$ we have $S_{H \cdot x} = \{[H \cdot x, s] : s \in H\}$. We can define a continuous surjection $\phi : H \to S_{H \cdot x}$ by $\phi(s) = [H \cdot x, s]$ and it is clear from the definition of the operations on S that this is a homomorphism. Next, if $[H \cdot x, s_i] \to [H \cdot x, s]$ in $S_{H \cdot x}$ then we can use the fact that Q is open to pass to a subnet, relabel, and choose new

s_i such that $s_i \to s$. However, this implies that ϕ is an open map. Finally, it is clear from the definition of $\cong$ that $\phi(s) = \phi(t)$ if and only if $st^{-1} \in H_x$. It follows that ϕ factors to an isomorphism of H/H_x with $S_{H\cdot x}$. Since H/H_x is clearly abelian this proves that S has abelian fibres and we are done. $\square$

The reason we went through all of this rigmarole is that given a locally σ-trivial system (H, X) we would like to show that X is a principal $S_{(H,X)}$ bundle.

Proposition 2.27. *Suppose H is an abelian locally compact Hausdorff group acting on a locally compact Hausdorff space X with continuously varying stabilizers such that X/H is Hausdorff. If X is locally σ-trivial then X is a principal $S_{(H,X)}$-bundle.*

Proof. Let $q : X \to X/H$ be the quotient map. We know from Definition 2.24 that if $x \in X$ then there is an H-invariant neighborhood U that is homeomorphic to $U/H \times H/\cong$. If we let $V = q(U) = U/H$ then V is an open neighborhood of $H \cdot x$ and $q^{-1}(V) = U$. Furthermore, let p be the bundle map for S and observe that

$$p^{-1}(V) = \{[H \cdot x, s] \in S : H \cdot x \in V\} = U/H \times H/\cong .$$

Thus, as per Definition 2.24, there is a homeomorphism $\phi_V : q^{-1}(V) \to p^{-1}(V)$. Furthermore, since $\phi_V(x) = [H \cdot x, s]$ for some $s \in H$ it is clear that $q = p \circ \phi_V$. Now, find one of these neighborhoods for every $x \in X$ and use them to build an open cover $\mathcal{V}$ of X/H.

For V_i in this open cover let $\phi_i = \phi_{V_i}$ and note that we have already shown that each ϕ_i is a bundle map. Given V_{ij} define $\gamma_{ij} : U_{ij} \to S$ by $\gamma_{ij}(H\cdot x) = \phi_i \circ \phi_j^{-1}([H\cdot x, e])$. It is clear that γ_{ij} is a continuous section on V_{ij}. Since γ_{ij} is a section we can find a function $\tilde{\gamma}_{ij}$ from V_{ij} into H such that $\gamma_{ij}(H \cdot x) = [H \cdot x, \tilde{\gamma}_{ij}(H \cdot x)]$. Suppose $[G \cdot x, s] \in p^{-1}(V_{ij})$. Then, using the equivariance condition of Definition 2.24, we have

$$\begin{aligned} \phi_i \circ \phi_j^{-1}([H \cdot x, s]) &= \phi_i(s \cdot \phi_j^{-1}([H \cdot x, e])) \\ &= [H \cdot x, \tilde{\gamma}_{ij}(H \cdot x)s] \\ &= \gamma_{ij}(H \cdot x)[H \cdot x, s]. \end{aligned}$$

Thus X is a principal S-bundle with trivialization $(\mathcal{V}, \phi, \gamma)$. $\square$

Remark 2.28. In [RW88] σ-trivial space is said to be locally liftable if every continuous section $c : X/H \to S$ is given locally by a continuous map $\tilde{c} : X/H \to H$ such that $c(H \cdot x) = [H \cdot x, \tilde{c}(H \cdot x)]$. Locally σ-trivial *bundles* are defined to be locally σ-trivial spaces which are also locally liftable. The reason for this extra requirement has to do with finding a cohomological invariant for X. Let $\mathcal{T}$ be the sheaf defined for $U \subset X/H$ by $\mathcal{T}(U) = C(U, H)$ and $\mathcal{R}$ be the subsheaf of $\mathcal{T}$ where $\mathcal{R}(U)$ is the subset

of $\mathcal{T}(U)$ such that $f(H \cdot x) \in H_x$ for all x. Then sheaf cohomological considerations will show that we can construct a quotient sheaf $\mathcal{T}/\mathcal{R}$ and an associated cohomology $H^n(X/H;\mathcal{T}/\mathcal{R})$. Given a σ-trivial space one would like to use the transition maps γ_{ij}, as defined in the proof of Proposition 2.27, to construct an element of $H^1(X/H;\mathcal{T}/\mathcal{R})$. The problem is that while γ_{ij} is a continuous section of U_{ij} into S the associated map $\tilde{\gamma}_{ij} : U_{ij} \to H$ may not be continuous. If $\tilde{\gamma}_{ij}$ is not continuous then it doesn't define an element of $\mathcal{T}(U_{ij})$ and we cannot construct the appropriate cohomology element. However, if X is required to be locally liftable then, by passing to a smaller open set, we can guarantee that $\tilde{\gamma}_{ij}$ is a continuous function. As such it defines an element of $\mathcal{T}(U_{ij})$ and hence a cohomology element in $H^1(X/H;\mathcal{T}/\mathcal{R})$. In fact, it is shown in [RW88] that this construction leads to a one-to-one correspondence between locally σ-trivial *bundles* with a fixed orbit space X/H and $H^1(X/H;\mathcal{T}/\mathcal{R})$.

This is an artificial restriction in our setting. The γ_{ij} can always be used to define an element of $H^1(S_{(X,H)})$, regardless of whether σ is locally liftable or not. It is comforting to observe the following however. Let $\mathcal{S}$ be the sheaf of local sections of S so that $H^n(S) = H^n(X/H;\mathcal{S})$ by definition. It is straightforward to show that if σ is locally liftable then we get a short exact sequence of sheaves

$$0 \to \mathcal{R} \to \mathcal{T} \to \mathcal{S} \to 0$$

and that $H^n(X/H;\mathcal{S})$ is naturally isomorphic to $H^n(X/H,\mathcal{T}/\mathcal{R})$. Furthermore, once one sorts out all of the various constructions, it is clear that the different cohomological invariants of a σ-trivial bundle are identified under this isomorphism.

One reason for preferring the locally liftable case is that the cohomology group $H^1(X/H;\mathcal{T}/\mathcal{R})$ is easier to deal with. This is because it is a quotient of $H^1(X/H;\mathcal{T})$ and $H^1(X/H;\mathcal{T})$ is just the classical sheaf cohomology of X/H with coefficients in H. In general the cohomology group $H^1(S_{(X,H)})$ is much more mysterious. For instance, it is not immediately clear that there are group bundles S such that $H^1(S)$ is nontrivial. However, any nontrivial example of a locally σ-trivial space will give rise to a nontrivial principal S-bundle. This will in turn guarantee that the cohomology group $H^1(S)$ is nontrivial.

The following examples are lifted straight from [RW88] and are included for completeness.

Example 2.29 ([RW88, Example 4.6]). Choose a complex line bundle $p : L \to Y$, and give it a Hermitian structure. Then we can define an action of $H = \mathbb{R}$ on L by

$$r \cdot z = \begin{cases} e^{2\pi i r/|z|} \cdot z & \text{if } |z| \neq 0, \\ z & \text{if } |z| = 0. \end{cases}$$

Then we have
$$H_z = \begin{cases} |z|\mathbb{Z} & \text{if } |z| \neq 0, \\ \mathbb{R} & \text{if } |z| = 0. \end{cases}$$
It is fairly easy to see that the stabilizers vary continuously. The local triviality of L as a bundle implies that there are local cross sections of p, so that by [RW88, Proposition 4.3] L is a locally σ-trivial space. In general, L is not globally σ-trivial. In fact, it is easy to see that it is globally σ-trivial exactly when it is trivial as a line bundle. Thus, if L is a non-trivial line bundle then L is a non-trivial principal S-bundle where $S = (L/\mathbb{R} \times \mathbb{R})/\cong$.

Example 2.30 ([RW88, Example 4.15]). Let H be a locally compact abelian group, $q : Y \to Z$ a locally trivial principal H-bundle, and $\tau : Z \to \mathbb{RP}^n$ a continuous map onto n-dimensional real projective space. Now define $X = Y \times \mathbb{R}^{n+1}/\sim$ where $(x, a) \sim (y, b)$ if and only if $x = y$ and $a - b \in \tau(x)$. Let $K = H \times \mathbb{R}^{n+1}$ act on X by $(s, a) \cdot (y, b) := (s \cdot y, a + b)$. Then it is shown in [RW88, Example 4.15] that X is a locally σ-trivial bundle over Z which is globally σ-trivial if and only if Y is trivial. It follows that X is a principal $S_{(K,X)}$-bundle. For an interesting concrete example of such a space we can take $H = \mathbb{Z}_2$, $Y = S^n$, $Z = \mathbb{RP}^n$, $q : Y \to Z$ the canonical map, and τ the identity map.

2.2 Group Bundle Duality

The goal of this section is to show that when S is an abelian, continuously varying, locally compact Hausdorff group bundle then S has a Pontryagin duality theory which parallels the duality theory of abelian locally compact Hausdorff groups. We start by recalling the basic facts concerning the group case.

Remark 2.31. If H is an abelian locally compact Hausdorff group then the Pontryagin dual of H, denoted $\widehat{H}$, is defined to be the set of continuous $\mathbb{T}$-valued homomorphisms on H. Elements of $\widehat{H}$ are called characters and $\widehat{H}$ is an abelian group under the operations of pointwise multiplication and conjugation. Furthermore, the topology of uniform convergence on compact sets makes $\widehat{H}$ into an abelian locally compact Hausdorff group [Rud62]. Recall that for second countable spaces, the topology of uniform convergence on compacta is characterized by $\omega_i \to \omega$ if and only if $\omega_i(s_i) \to \omega(s)$ for all $s_i \to s$ [Wil07, Lemma 1.30].

We can also realize $\widehat{H}$ as the spectrum of the group C^*-algebra in the following way. Those readers unfamiliar with group C^*-algebras are referred to [RW98, Appendix C.3]. First, observe that $C^*(H)$ is abelian since H is. Given a character $\omega \in \widehat{H}$ we

define a function on $C_c(H)$, also denoted ω, by

$$\omega(f) = \hat{f}(\omega) := \int_H f(s)\omega(s)ds. \tag{2.3}$$

Then ω extends to a character on $C^*(G)$. Furthermore this character is uniquely determined by ω and every element of the spectrum can be obtained in this fashion. Whats more, the topology of uniform convergence on compacta on $\widehat{H}$ is exactly the Gelfand topology when $\widehat{H}$ is viewed as the spectrum of $C^*(H)$.

Given a function $f \in C_c(H)$ the function $\hat{f}$ defined in (2.3) by

$$\omega \mapsto \omega(f) = \int_H f(s)\omega(s)ds$$

is called the Fourier transform of f. The driving result behind harmonic analysis is that the map defined on $C_c(H)$ via $f \mapsto \hat{f}$ extends to an isomorphism from $C^*(H)$ onto $C_0(\widehat{H})$. Of course, this is a special case of the usual Gelfand-Naimark theorem for abelian C^*-algebras. This discussion has been a short version of [Wil07, Section 3.1].

The basic idea is that given an abelian group bundle S we just "bundle together" the duals of the fibres of S. The question is what to use as the topology on the total space.

Definition 2.32. Suppose S is an abelian, continuously varying, locally compact Hausdorff group bundle with bundle map p. We define the *dual bundle* of S to be the disjoint union

$$\widehat{S} := \coprod_{u \in S^{(0)}} \widehat{S}_u$$

where $\widehat{S}_u$ denotes the Pontryagin dual of S_u. We define the *dual bundle map* $\hat{p} : \widehat{S} \to S^{(0)}$ to be given for $\omega \in \widehat{S}$ by $\hat{p}(\omega) = u$ if $\omega \in \widehat{S}_u$. If β is a Haar system for S then given $\omega \in \widehat{S}$ and $f \in C_c(S)$ we define

$$\omega(f) := \int_S f(s)\omega(s)d\beta^{\hat{p}(\omega)}(s). \tag{2.4}$$

Finally, we define a topology on $\widehat{S}$ as follows. Given a net $\{\omega_i\} \subset \widehat{S}$ and $\omega \in \widehat{S}$ then $\omega_i \to \omega$ if and only if $\omega_i(f) \to \omega(f)$ for all $f \in C_c(S)$.

Remark 2.33. Defining a topology by specifying the convergent sequences can be a subtle process. We will show that in this case these convergent sequences characterize the Gelfand topology on $\widehat{S}$ as the spectrum of the groupoid C^*-algebra of S.

Remark 2.34. Since $\hat{p} : \widehat{S} \to S^{(0)}$ is a surjection we can view $\widehat{S}$ as a bundle over $S^{(0)}$ with fibres $\hat{p}^{-1}(u)$. However, by construction, $\hat{p}^{-1}(u)$ is clearly equal to the dual of S_u which lies inside $\widehat{S}$. Therefore the notation $\widehat{S}_u$ can be used to denote both the dual of S_u and the fibre of $\widehat{S}$ over u without confusion.

Example 2.35. Before checking the details of Definition 2.32 below, it is worth pointing out that if Z and T are the bundles described in Example 1.43 then Z is isomorphic to $\widehat{T}$ and T is isomorphic to $\widehat{Z}$.

There are a lot of assertions in Definition 2.32 that deserve to be checked and unfortunately we lack the technology to do them all justice at the moment. Fortunately, these issues are worked through in [PSMW96]. We will be developing the necessary technology later on, however, and for those readers not opposed to a little forward referencing a proof to the following lemma is provided. Basically, we need a little operator algebra theory to show that the topology on $\widehat{S}$ is well behaved.

Lemma 2.36. *Suppose S is an abelian, continuously varying, second countable, locally compact Hausdorff groupoid. Then the dual bundle $\widehat{S}$ is a second countable locally compact Hausdorff space. Furthermore, the map $\hat{p} : \widehat{S} \to S^{(0)}$ is a continuous surjection, and $\hat{p}^{-1}(u)$ may be identified with the dual of S_u topologically.*

Proof. These statements are proved in the discussion at the beginning of [PSMW96, Section 3]. An explicit proof, that unfortunately relies on some material in Section 4.3, is given below.

Since S is continuously varying it has a Haar system β. Therefore we can construct the groupoid C^*-algebra $C^*(S)$. Furthermore, it is easy to see that, because S is abelian, $C^*(S)$ is abelian also. It then follows from the Gelfand-Naimark theorem [Arv76] that given the Gelfand topology the spectrum of $C^*(S)$ is a locally compact Hausdorff space. Next, Proposition 4.34 implies that $C^*(S)$ is a $C_0(S^{(0)})$-algebra and that given $u \in S^{(0)}$ the fibre $C^*(S)(u)$ is (isomorphic to) $C^*(S_u)$. It is a result of general $C_0(X)$-algebra theory, reproduced in Section 3.1.1 as Proposition 3.22, that there is a continuous surjection $\hat{p} : C^*(S)^\wedge \to S^{(0)}$ such that if $\pi \in C^*(S)^\wedge$ then π factors to an irreducible representation of $C^*(S)(\hat{p}(\pi)) = C^*(S_{\hat{p}(\pi)})$. In this way we can identify $C^*(S)^\wedge$ with the disjoint union $\coprod_{u \in S^{(0)}} C^*(S_u)^\wedge$. However, we know from Remark 2.31 that $C^*(S_u)^\wedge = \widehat{S}_u$. It follows that $\widehat{S}$ can be identified with $C^*(S)^\wedge$ as a set and that $\hat{p}$ has exactly the right definition under this identification.

Now, it is also a result of Proposition 4.34 that the quotient map from $C^*(S)$ to $C^*(S)(u) = C^*(S_u)$ is given on $C_c(S)$ by restriction to S_u. Suppose $\omega \in \widehat{S}$ and let $u = \hat{p}(\omega)$. Then we can lift ω to a representation of $C^*(S)$ and for $f \in C_c(S)$ this is given by

$$\omega(f) = \int_{S_u} f|_{S_u}(s)\omega(s)d\beta^u(s) = \int_S f(s)\omega(s)d\beta^u(s).$$

Therefore the action of ω on $C_c(S)$ defined in (2.4) is precisely the action of ω on $C_c(S)$ as an element of $C^*(S)^\wedge$. Since the Gelfand topology on $C^*(S)^\wedge$ is characterized by pointwise convergence [Arv76], we conclude that the topology on $\widehat{S}$ defined in Definition 2.32 is exactly the Gelfand topology when $\widehat{S}$ is identified with $C^*(S)^\wedge$. Thus $\widehat{S}$ is locally compact Hausdorff. Furthermore, it immediately follows that the map $\hat{p}$ is continuous as a function on $\widehat{S}$, and that the topology on $\widehat{p}^{-1}(u)$ is the Gelfand topology on $C^*(S_u)^\wedge$, which in turn is the topology on the dual of S_u. Finally, it follows from Corollary 4.15 that $C^*(S)$ is separable. Hence $C^*(S)^\wedge$ is second countable [Dix77, Proposition 3.3.4] and we are done □

Remark 2.37. It follows from the proof of Proposition 2.36 that $\widehat{S}$ can be identified with the spectrum of $C^*(S)$ so that characters act on elements of $C_c(S)$ as in (2.4). However, we can then use the Gelfand-Naimark theorem to conclude that the Fourier transform induces an isomorphism of $C^*(S)$ with $C_0(\widehat{S})$. In this way bundle duality is a generalization of the usual Pontryagin duality for groups. It is notable that the dual of S is often *defined* to be the spectrum of the C^*-algebra of S.

The following proposition is important because it gives an alternative sequential characterization of the topology on $\widehat{S}$. This proposition, and its proof, can be found in [PSMW96, Proposition 3.3] and are only reproduced here for ease of reference. In particular, the argument used in this proof will be used again and again later on.

Proposition 2.38. *Suppose S is an abelian, continuously varying, second countable, locally compact Hausdorff group bundle. A sequence $\{\omega_i\}$ in $\widehat{S}$ converges to ω_0 in $\widehat{S}$ if and only if*

(a) *$\hat{p}(\omega_i) \to \hat{p}(\omega_0)$ in $S^{(0)}$, and*

(b) *if $s_i \in S_{\hat{p}(\omega_i)}$ for all $i \geq 0$ and $s_i \to s$ in S, then $\omega_i(s_i) \to \omega_0(s_0)$.*

Proof. First, suppose that ω_i converges to ω_0 and let $u_i = \hat{p}(\omega_i)$ for all $i \geq 0$. The continuity of $\hat{p}$ implies that $u_i \to u_0$. If condition (b) fails then there are $s_i \in S_{u_i}$ converging to $s_0 \in S_{u_0}$ with $\omega_i(s_i)$ not converging to $\omega_0(s_0)$. Clearly, we may assume that no subsequence converges to $\omega_0(s_0)$ either. Next, we observe that we may assume $u_i \neq u_0$ for all i; otherwise we obtain an immediate contradiction by passing to a subsequence and relabeling so that $u_i = u_0$ for all i, and $\omega_i \to \omega_0$ in $\widehat{S}_{u_0}$. Furthermore, again passing to a subsequence and relabeling, we can assume that $u_i \neq u_j$ if $i \neq j$. In particular, we can define an integer valued function on $C = p^{-1}(\{u_i\}_{i\geq 0})$ by $\iota(s) = i$ when $p(s) = u_i$. Now fix $f \in C_c(S)$ with $\omega_0(f) = 1$. Notice that C is closed, and $g_0 : C \to \mathbb{C}$, defined by

$$g_0(t) = f(s_{\iota(t)}^{-1}t) \quad \text{for } t \in C,$$

is continuous and compactly supported. The Tietze Extension Theorem implies that there is a $g \in C_c(S)$ extending g_0. But

$$\omega_i(g) = \omega_i(s_i)\omega_i(f)$$

for $i \geq 0$. We obtain the desired contradiction by noting that $\omega_i(f) \to 1$ and $\omega_i(g) \to \omega_0(s_0)$.

Conversely, now assume that ω_i satisfies conditions (a) and (b). Let $u_i = \hat{p}(\omega_i)$ for all $i \geq 0$. Suppose there is a $f \in C_c(S)$ such that $\omega_i(f)$ fails to converge to $\omega_0(f)$. As above we can reduce to the case that $u_i \neq u_j$ if $i \neq j$. This time we define $g_0 : C \to \mathbb{C}$ by

$$g_0(t) = \omega_{\iota(t)}(t)f(t) \quad \text{for } t \in C.$$

Again a few moments of reflection reveal that g_0 is continuous and compactly supported so that there is a $g \in C_c(S)$ extending g_0. The continuity of the Haar system on S implies that

$$\int_S g(s)d\beta^{u_i}(s) \to \int_S g(s)d\beta^{u_0}(s).$$

Since $\int_S g d\beta^{u_i} = \omega_i(f)$ for all $i \geq 0$ we obtain the necessary contradiction. $\square$

This characterization of the topology on $\widehat{S}$ is so nice that we will restrict ourselves to the second countable case for the rest of the section. The following proposition is, more or less, [PSMW96, Corollary 3.4].

Proposition 2.39. *Suppose S is an abelian, continuously varying, second countable locally compact Hausdorff group bundle. Then the dual bundle $\widehat{S}$ is an abelian, second countable, locally compact Hausdorff group bundle where $\widehat{S}^{(2)} = \{(\omega, \chi) \in \widehat{S} \times \widehat{S} : \hat{p}(\omega) = \hat{p}(\chi)\}$ and the operations are given pointwise by*

$$\omega\chi(s) := \omega(s)\chi(s) \qquad\qquad \omega^{-1}(s) := \overline{\omega(s)}$$

The unit space $\widehat{S}^{(0)}$ can be identified with $S^{(0)}$ and under this identification the bundle map for $\widehat{S}$ is $\hat{p}$. Furthermore, the fibres of $\widehat{S}$ are the Pontryagin duals of the fibres of S. Finally, the topology on $\widehat{S}$ is independent of the choice of Haar measure.

Proof. We proved in Lemma 2.36 that $\widehat{S}$ is a second countable locally compact Hausdorff space. Furthermore, it is clear from the definition of the dual bundle that the fibres of $\widehat{S}$ are the Pontryagin duals of the fibres of S set theoretically. It follows from Lemma 2.36 that the fibres of $\widehat{S}$ can be identified with the Pontryagin duals of the fibres of S topologically. Finally, since we have characterized the topology on $\widehat{S}$ independently of Haar measure in Proposition 2.38, it follows that the topology is independent of Haar measure.

Next, we have defined the operations on $\widehat{S}$ fibrewise by the usual pointwise operations on the Pontryagin duals. Since the dual of a group is again a group it is easy to see that $\widehat{S}$ is a groupoid. We would like to see that the operations are continuous. Suppose $\omega_i \to \omega$ in $\widehat{S}$. Let $s_i \to s$ such that $p(s_i) = \hat{p}(\omega_i)$ and $p(s) = \hat{p}(\omega)$. Then, by Proposition 2.38 $\omega_i(s_i) \to \omega(s)$ and therefore

$$\omega_i^{-1}(s_i) = \overline{\omega_i(s_i)} \to \overline{\omega(s)} = \omega^{-1}(s).$$

It follows from Proposition 2.38 that this implies $\omega_i^{-1} \to \omega$. The proof that multiplication is continuous is similar.

We constructed $\widehat{S}$ so that the range and source maps are both given by $\omega \mapsto \omega^{-1}\omega$. However, $\omega^{-1}\omega = 1_u$ where 1_u denotes the trivial character on S_u. We would like to identify $S^{(0)}$ and $\widehat{S}^{(0)}$ via the map $u \mapsto 1_u$. It is easy to see that this map is a bijection. Proposition 2.38 implies that if $1_{u_i} \to 1_u$ then $u_i \to u$ and it is easy to see that Proposition 2.38 also implies that $1_{u_i} \to 1_u$ if $u_i \to u$. Therefore we can topologically identify $S^{(0)}$ and $\widehat{S}^{(0)}$ and clearly the bundle map on $\widehat{S}$ is $\hat{p}$ under this identification. Finally, since the dual of an abelian group is abelian it follows that $\widehat{S}$ is an abelian group bundle. □

At this point we need to recall some basic facts about abelian harmonic analysis. These theorems can, for the most part be found in [Rud62, Chapter 1]. Those portions of the theorems which are not explicitly proved in [Rud62] are proved here. It is notable that [Rud62] uses the opposite conjugation convention and that the following theorems have been modified accordingly.

Definition 2.40. Let H be a locally compact Hausdorff abelian group. A function f defined on H is said to be *positive definite* if the inequality

$$\sum_{i,j=1}^{N} c_i \overline{c_j} f(s_n s_m^{-1}) \geq 0$$

holds for every choice of $s_1, \ldots, s_N \in H$ and for every choice of $c_1, \ldots, c_N \in \mathbb{C}$.

Remark 2.41. Definition 2.40 is a special case of Definition 1.105 in that if we let H act on the trivial space consisting of one point then the positive definite functions are exactly those of positive type.

Example 2.42. Suppose H is a locally compact abelian group and $g \in L^2(H)$ where H is given Haar measure. For $s \in H$ define

$$f(s) = g^* * g(s) := \int \overline{g(t^{-1})} g(t^{-1}s)dt.$$

It follows from [Rud62, 1.4.2] that f is continuous and positive definite.

This the premiere example of a positive definite function and are the only positive definite functions that we will be using. The next theorem says that positive definite functions are all given by integration with respect to some measure on $\widehat{H}$.

Theorem 2.43 (Bochner's Theorem [Rud62, 1.4.3])**.** *Let H be an abelian locally compact Hausdorff group. A continuous function f on H is positive definite if and only if there is a finite non-negative measure μ on $\widehat{H}$ such that for $s \in H$*

$$f(s) = \int_{\widehat{H}} \overline{\omega(s)} d\mu(\omega). \tag{2.5}$$

Furthermore $\mu(\widehat{H}) = \|f\|_\infty$.

Proof. The first part of the theorem is Bochner's Theorem as stated in [Rud62]. For the last statement, since f is positive definite, $|f(s)| \leq f(0)$ for all $s \in H$ [Rud62, 1.4.1]. Hence

$$\|f\|_\infty = f(0) = \int_{\widehat{H}} \overline{\omega(0)} d\mu(\omega) = \mu(\widehat{H}). \qquad \square$$

It is not particularly hard to see that the span of the functions of positive type are exactly those functions defined via (2.5) except where μ is a complex measure. The following theorem says that the Fourier transform is very well behaved for this kind of function.

Theorem 2.44 (Inversion Theorem [Rud62, 1.5.1])**.** *Let H be an abelian locally compact Hausdorff group with Haar measure λ. Suppose $f \in L^1(H)$ is such that for all $s \in H$*

$$f(s) = \int_{\widehat{H}} \overline{\omega(s)} d\mu(\omega)$$

for some complex measure μ on $\widehat{H}$. Then the Fourier transform $\hat{f}$ (defined in Remark 2.31) is in $L^1(\widehat{H})$. Furthermore, there is a Haar measure $\widehat{\lambda}$ on $\widehat{H}$ such that for all functions of this form

$$f(s) = \int_{\widehat{H}} \hat{f}(\omega)\overline{\omega(s)} d\widehat{\lambda}(\omega).$$

Remark 2.45. Given $f \in L^1(\widehat{H})$ the function

$$\check{f}(s) = \int_{\widehat{H}} f(\omega)\overline{\omega(s)} d\widehat{\lambda}(\omega)$$

is called the "inverse Fourier transform" of f. Colloquially Theorem 2.44 says that for a certain class of functions on H the inverse Fourier transform of the Fourier transform of f is equal to f.

Definition 2.46. Given an abelian locally compact Hausdorff group H with Haar measure λ we call the measure $\widehat{\lambda}$ coming from Theorem 2.44 the *dual Haar measure*. We generally denote integration with respect to λ by ds and integration with respect to $\widehat{\lambda}$ by $d\omega$.

Theorem 2.47 (Plancharel's Theorem [Rud62, 1.6.1])**.** *Let H be an abelian locally compact Hausdorff group with Haar measure λ and dual measure $\widehat{\lambda}$. The Fourier transform, restricted to $L^1(H) \cap L^2(H)$, is an isometry with respect to the L^2-norms on $L^2(H, \lambda)$ and $L^2(\widehat{H}, \widehat{\lambda})$. Furthermore it maps onto a dense subspace of $L^2(\widehat{H})$ and can be extended to an isometry of $L^2(H)$ onto $L^2(\widehat{H})$.*

It is notable that the isomorphism between $L^2(H)$ and $L^2(\widehat{H})$ given by Theorem 2.47 cannot be explicitly defined off of $L^1(H) \cap L^2(H)$.

Lemma 2.48. *If H and f are as in Theorem 2.44 then $d\mu = \hat{f} d\omega$.*

Proof. For $g \in C_c(H)$ we have, by Theorem 2.47,

$$\begin{aligned} \int_{\widehat{H}} \overline{\hat{g}} \hat{f} d\omega &= \int_H \overline{g} f ds = \int_H \int_{\widehat{H}} \overline{g(s)\omega(s)} d\mu(\omega) ds \qquad (2.6) \\ &= \int_{\widehat{H}} \overline{\int_G g(s)\omega(s) ds} d\mu(\omega) = \int_{\widehat{H}} \overline{\hat{g}} d\mu(\omega). \end{aligned}$$

It is well known [Rud62] that the image of $C_c(H)$ under the Fourier transform is dense in $C_0(\widehat{H})$. Given $\epsilon > 0$ and $\phi \in C_0(\widehat{H})$ choose $g \in C_c(H)$ such that

$$\|\hat{g} - \phi\|_\infty < \frac{\epsilon}{2 \max\{|\mu|(\widehat{H}), \|\hat{f}\|_1\}}.$$

Then, using (2.6), we have

$$\begin{aligned} \left| \int_{\widehat{H}} \overline{\phi} \hat{f} d\omega - \int_{\widehat{H}} \overline{\phi} d\mu \right| &\leq \left| \int_{\widehat{H}} (\overline{\phi} - \overline{\hat{g}}) \hat{f} d\omega \right| + \left| \int_{\widehat{H}} (\overline{\phi} - \overline{\hat{g}}) d\mu \right| \\ &\leq \int_{\widehat{H}} |\phi - \hat{g}| |\hat{f}| d\omega + \int_{\widehat{H}} |\phi - \hat{g}| d|\mu| \\ &\leq \|\phi - \hat{g}\|_\infty (\|\hat{f}\|_1 + |\mu|(\widehat{H})) < \epsilon. \end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$ and $\phi \in C_0(\widehat{H})$, after replacing ϕ by $\overline{\phi}$ we may conclude that $\hat{f} d\omega = d\mu$. $\square$

At this point we have all the classical harmonic analysis that we need. It is time to return to the groupoid case. First, we would like to see that given a continuously

varying abelian group bundle the dual bundle is also continuously varying. Fortunately, this very thing is shown in [PSMW96], once we recall that a group bundle has a Haar system if and only if it is continuously varying. We have reproduced the proof here for completeness.

Proposition 2.49 ([PSMW96, Proposition 3.6]). *If S is an abelian, second countable, locally compact Hausdorff group bundle with Haar system $\beta = \{\beta^u\}$ then the dual measures $\hat{\beta}^u$ form a Haar system for the dual bundle $\widehat{S}$.*

Proof. Suppose that K is compact in $\widehat{S}$. We claim that $u \mapsto \hat{\beta}^u(K)$ is bounded on $S^{(0)}$. Of course, it suffices to consider only $u \in \hat{p}(K)$. Let $f \in C_c(S)$ be a non-negative function such that

$$\int_{S_u} f(s)^2 d\beta^u(s) = 1 \text{ for all } u \in \hat{p}(K). \tag{2.7}$$

Since $\hat{p}(K)$ is compact, there is an $\epsilon > 0$ so that

$$\int_{S_u} f(s) d\beta^u(s) > \epsilon \text{ for all } u \in \hat{p}(K). \tag{2.8}$$

Using Theorem 2.47 (2.7) implies that

$$\int_{\widehat{S}_u} |\hat{f}(\omega)|^2 d\hat{\beta}^u(\omega) = 1 \text{ for all } u \in \hat{p}(K). \tag{2.9}$$

Moreover the continuity of $\hat{f}$ and (2.8) imply that $U = \{\omega \in \widehat{S} : |\hat{f}(\omega)|^2 > \epsilon^2\}$ is an open neighborhood of $\hat{p}(K)$.

If $\omega \in K$, then $\omega^{-1}\omega \in \hat{p}(K)$. The continuity of multiplication implies that there is a neighborhood V of ω such that $V^{-1}V \subset U$. Therefore there is a cover $V_1, \ldots, V_m$ of K such that $V_j^{-1}V_j \subset U$ for each $1 \leq j \leq m$. In view of (2.9), $\hat{\beta}^u(U) \leq \epsilon^{-2}$ if $u \in \hat{p}(K)$. Furthermore, if $\hat{\beta}^u(V_j) \neq 0$, then there is an $\omega \in V_j$ with $\hat{p}(\omega) = u$. Then

$$\hat{\beta}^u(V_j) = \hat{\beta}^u(\omega\omega^{-1}V_j) \leq \hat{\beta}^u(\omega V_j^{-1}V_j) \leq \hat{\beta}^u(\omega U) = \hat{\beta}^u(U) \leq 1/\epsilon^2.$$

It follows that $\hat{\beta}^u(K) \leq m/\epsilon^2$ for all $u \in S^{(0)}$. This proves the claim.

Now let $\{u_i\}_{i\in I}$ be a net in $S^{(0)}$ converging to $u \in S^{(0)}$. If $\phi \in C_c(\widehat{S})$, then let $\hat{\beta}(\phi)(v) = \int_{\hat{S}_v} \phi(\omega) d\hat{\beta}^v(\omega)$. The above argument implies that $\{\hat{\beta}(\phi)(u_i)\}_{i\in I}$ is bounded. Thus if ω is a generalized limit[1] on $\ell^\infty(I)$, then we obtain a positive linear functional μ on $C_c(\widehat{S})$ by $\mu(\phi) = \omega(\{\hat{\beta}(\phi)(u_i)\})$. Suppose that $\phi, \psi \in C_c(\widehat{S})$ agree on

[1] A generalized limit is a norm one extension of the ordinary limit functional on the subspace c_0 of $\ell^\infty(I)$ consisting of those nets $\{a_i\}$ such that $\lim_i a_i$ exists.

$\widehat{S}_u$. Then if K is a compact set containing the supports of ϕ and ψ,

$$\begin{aligned}|\hat{\beta}(\phi)(u_i) - \hat{\beta}(\psi)(u_i)| &\leq \int_{\widehat{S}_{u_i}} |\phi(\omega) - \psi(\omega)| d\hat{\beta}^{u_i}(\omega) \\ &\leq \sup_{\omega \in \widehat{S}_{u_i}} |\phi(\omega) - \psi(\omega)| \hat{\beta}^{u_i}(K).\end{aligned}$$

Since $\sup_{\omega \in \widehat{S}_v} |\phi(\omega) - \psi(\omega)|$ tends to zero as v tends to u and since $v \mapsto \hat{\beta}^v(K)$ is bounded, it follows that $\mu(\phi) = \mu(\psi)$. Since every function in $C_c(\widehat{S}_u)$ has an extension to an element of $C_c(\widehat{S})$, we can view μ as a Radon measure on $\widehat{S}_u$.

However, if $f \in C_c(S)$ then, by the Plancharel Theorem, $\hat{\beta}(|\hat{f}|^2)(u_i) = \beta(|f|^2)(u_i)$ which converges to $\beta(|f|^2)(u) = \hat{\beta}(|\hat{f}|^2)(u)$. It follows that

$$\hat{\beta}(|\hat{f}|^2)(u) = \mu(|\hat{f}|^2) \text{ for all } f \in C_c(S_u). \tag{2.10}$$

By density, (2.10) holds for all $\hat{f} \in L^2(\widehat{S}_u, \hat{\beta}^u)$. In particular $\mu = \hat{\beta}^u$ on $C_c(\widehat{S}_u)$.

We have shown that if $\{u_i\}$ is any net converging to u in $S^{(0)}$, then $\omega(\{\hat{\beta}(\phi)(u_i)\}) = \hat{\beta}(\phi)(u)$. Therefore $\lim_i \hat{\beta}(\phi)(u_i) = \hat{\beta}(\phi)(u)$, and it follows that $\{\hat{\beta}^u\}$ is a Haar system. □

Definition 2.50. Given an abelian, second countable locally compact Hausdorff group bundle with Haar system $\beta = \{\beta^u\}$ then the Haar system formed by the collection of dual measures $\hat{\beta} = \{\hat{\beta}^u\}$ is called the *dual Haar system*.

This is interesting because it means that given an abelian continuously varying group bundle S we can form the double dual $\widehat{\widehat{S}}$ by taking the dual of $\widehat{S}$. It is natural to ask whether or not this is isomorphic to the original group bundle. The following lemma gets us most of the way there.

Lemma 2.51. *Given an abelian continuously varying group bundle S there is a continuous bijective groupoid homomorphism $\Phi : S \to \widehat{\widehat{S}}$ given for $s \in S$ and $\omega \in \widehat{S}$ by*

$$\Phi(s)(\omega) = \hat{s}(\omega) := \omega(s). \tag{2.11}$$

Proof. It follows from Proposition 2.39 that $\widehat{\widehat{S}}_u$ is the double dual of S_u for all $u \in S^{(0)}$. Furthermore, classical Pontryagin duality [Rud62] states that $s \mapsto \hat{s}$ is a group isomorphism from S_u onto $\widehat{\widehat{S}}_u$ for all $u \in S^{(0)}$. Since Φ is formed by gluing all of these fibre isomorphisms together it is clear that Φ is at least a bijective groupoid homomorphism. Next, we show that it is continuous. Suppose $s_i \to s$ in S and let $u_i = p(s_i)$ and $u = p(s)$. By Proposition 2.38 it will suffice to show, one, that $\hat{p}(\Phi(s_i)) \to \hat{p}(\Phi(s))$ and, two, that given $\omega_i \in \widehat{S}_{u_i}$ and $\omega \in \widehat{S}_u$ such that $\omega_i \to$

ω then $\Phi(s_i)(\omega_i) \to \Phi(s)(\omega)$. Since Φ preserves the bundle maps (it's a groupoid homomorphism) we know that $\hat{\hat{p}}(\Phi(s_i)) = u_i$ and $\hat{\hat{p}}(\Phi(s)) = u$ so that clearly the first condition holds. Now suppose $\omega_i \in \widehat{S}_{u_i}$ and $\omega \in \widehat{S}$ are such that $\omega_i \to \omega$. All we have to do is cite Proposition 2.38 yet again to see that

$$\Phi(s_i)(\omega_i) = \omega_i(s_i) \to \omega(s) = \Phi(s)(\omega). \qquad \square$$

If we were working with groups we would be done since continuous bijective group homomorphisms between second countable locally compact groups are automatically bicontinuous. This follows from Souslin's Theorem [Arv76, Theorem 3.2.3], which states that continuous bijections have a Borel inverse, and the fact that measurable homomorphisms between second countable locally compact groups are automatically continuous [Wil07, Theorem D.3]. However, we will show in Section 2.3 that this is "open mapping theorem" is not true for group bundles. Fortunately, it turns out that in this specific case Φ is a homeomorphism. It should be noted that the the following theorem is also stated, without proof, in [Ram98, Proposition 1.3.7].

Theorem 2.52 ([Goe09, Theorem 16])**.** *If S is an abelian, continuously varying, second countable, locally compact Hausdorff group bundle then the map $\Phi : S \to \widehat{\widehat{S}}$ such that $\Phi(s) = \hat{s}$ is a groupoid isomorphism.*

Proof. It follows from Lemma 2.51 that all we need to do is show that if $\{s_i\}_{i\geq 0} \subset S$ such that $\hat{s}_i \to \hat{s}_0$ then $s_i \to s_0$. Let $u_i = p(s_i)$ for all $i \geq 0$ and recall from (2.11) that $\hat{s}(\omega) := \omega(s)$. It follows from Definition 2.32 that $\hat{s}_i(\phi) \to \hat{s}_0(\phi)$ for all $\phi \in C_c(\widehat{S})$. Using (2.3) we have, for all $\phi \in C_c(\widehat{S})$,

$$\int_{\widehat{S}} \phi(\omega)\omega(s_i)d\hat{\beta}^{u_i}(\omega) \to \int_{\widehat{S}} \phi(\omega)\omega(s_0)d\hat{\beta}^{u_0}(\omega). \tag{2.12}$$

Now suppose we have a relatively compact open neighborhood V of u_0 in S. Then, using the continuity of the operations, there exists a relatively compact open neighborhood U of u_0 in S such that $U = U^{-1}$ and $U^2 \subset V$. Choose $h \in C_c(S)$ such that $h(u_0) = 1$ and $\operatorname{supp} h \subset U$. Define f on S by

$$f(s) := h^* * h(s) = \int_S \overline{h(t^{-1})}h(t^{-1}s)d\beta^{p(s)}(t).$$

It is straightforward to check that f is continuous. Furthermore, $f(s) = 0$ unless there exists $t \in S$ such that $t^{-1} \in U$ and $t^{-1}s \in U$. In other words, unless $s \in U^2 \subset V$. Therefore $\operatorname{supp} f \subset V$ and $f \in C_c(S)$. From now on let f^u denote the restriction of f to S_u. It is clear from the definition of f, and Example 2.42, that f^u is positive definite. Therefore f^u satisfies the conditions of Bochner's theorem and the inversion

theorem for all $u \in S^{(0)}$. In particular, this implies that for each $u \in S^{(0)}$ there exists a finite positive measure μ^u on $\widehat{S}_u S$, which we extend to $\widehat{S}$ by giving everything else measure zero, such that for all $s \in S$

$$f(s) = \int_{\widehat{S}} \overline{\omega(s)} \mu^{p(s)}(\omega).$$

Theorem 2.43 also implies that $\mu^u(\widehat{S}) = \mu^u(\widehat{S}_u) \leq \|f^u\|_\infty \leq \|f\|_\infty$ for all $u \in S^{(0)}$. It follows from Lemma 2.48 that for all $u \in S^{(0)}$.

$$\hat{f} d\hat{\beta}^u = d\mu^u \tag{2.13}$$

as measures on $\widehat{S}_u$. However, since both $\hat{\beta}^u$ and μ^u have support contained in $\widehat{S}_u$ equation (2.13) holds for $\hat{\beta}^u$ and μ^u as measures on all of $\widehat{S}$.

Now, we don't know that $\hat{f}$ is compactly supported. In fact, it's probably not. However if $\phi \in C_c(\widehat{S})$ then $\phi\hat{f}$ is compactly supported. It follows from (2.12) that

$$\int_{\widehat{S}} \phi(\omega)\hat{f}(\omega)\omega(s_i) d\hat{\beta}^{u_i}(\omega) \to \int_{\widehat{S}} \phi(\omega)\hat{f}(\omega)\omega(s_0) d\hat{\beta}^{u_0}(\omega). \tag{2.14}$$

Using (2.13) we can rewrite (2.14) as

$$\int_{\widehat{S}} \phi(\omega)\omega(s_i) d\mu^{u_i}(\omega) \to \int_{\widehat{S}} \phi(\omega)\omega(s_0) d\mu^{u_0}(\omega). \tag{2.15}$$

In order to make the notation a little more palatable we will temporarily define, for all $i \geq 0$,

$$a_i(\phi) := \int_{\widehat{S}} \phi(\omega)\omega(s_i) d\mu^{u_i}(\omega).$$

We would like to extend (2.15) to functions in $C_0(\widehat{S})$. Suppose we have $\psi \in C_0(\widehat{S})$ and are given $\epsilon > 0$. Choose $\phi \in C_c(\widehat{S})$ such that $\|\psi - \phi\|_\infty < \frac{\epsilon}{4\|f\|_\infty}$. Now choose I such that $|a_i(\phi) - a_0(\phi)| < \epsilon/2$ for all $i \geq I$. We then compute for $i \geq I$ that

$$\begin{aligned}
|a_i(\psi) - a_0(\psi)| &= \left| \int_{\widehat{S}} \psi(\omega)\omega(s_i) d\mu^{u_i} - \int_{\widehat{S}} \psi(\omega)\omega(s_0) d\mu^{u_0} \right| \\
&\leq \left| \int_{\widehat{S}} (\psi(\omega) - \phi(\omega))\omega(s_i) d\mu^{u_i} \right| \\
&\quad + \left| \int_{\widehat{S}} \phi(\omega)\omega(s_i) d\mu^{u_i} - \int_{\widehat{S}} \phi(\omega)\omega(s_0) d\mu^{u_0} \right| \\
&\quad + \left| \int_{\widehat{S}} (\psi(\omega) - \phi(\omega))\omega(s_0) d\mu^{u_0} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq |a_i(\phi) - a_0(\phi)| + \int_{\widehat{S}} |\psi - \phi| d\mu^{u_i} + \int_{\widehat{S}} |\psi - \phi| d\mu^{u_0} \\
&\leq |a_i(\phi) - a_0(\phi)| + (\mu^{u_i}(\widehat{S}) + \mu^{u_0}(\widehat{S}))\|\psi - \phi\|_\infty \\
&< \frac{\epsilon}{2} + \frac{2\|f\|_\infty \epsilon}{4\|f\|_\infty} = \epsilon.
\end{aligned}$$

It follows that if $\phi \in C_0(\widehat{S})$ then $a_i(\phi) \to a_0(\phi)$, or equivalently, that (2.14) holds for all $\phi \in C_0(\widehat{S})$.

Next, if $g \in C_c(S)$ then, since both g and f are compactly supported and β^x is regular for all $u \in S^{(0)}$, we have $f^u, g^u \in L^1(S_u, \beta^u) \cap L^2(S_u, \beta^u)$ for all $u \in S^{(0)}$. The Plancharel Theorem states that the Fourier transform is an isometry on $L^1(S_u) \cap L^2(S_u)$. Therefore we have, for all $u \in S^{(0)}$,

$$\int_{S_u} \overline{g^u} f^u d\beta^u = \int_{\widehat{S}_u} \overline{\widehat{g^u}} \widehat{f^u} d\hat{\beta}^u.$$

Observe that, for all $i \geq 0$ and $\omega \in \widehat{S}_{u_i}$,

$$\begin{aligned}
\overline{\widehat{g^{u_i}}(\omega)} \omega(s_i) &= \int_{S_{u_i}} \overline{g^{u_i}(s)\omega(s)} \omega(s_i) d\beta^{u_i}(s) = \int_{S_{u_i}} \overline{g^{u_i}(s)} \omega(s^{-1}s_i) d\beta^{u_i}(s) \\
&= \int_{S_{u_i}} \overline{g^{u_i}(s_i s)} \omega(s^{-1}) d\beta^{u_i}(s) = \overline{(\mathrm{lt}_{s_i^{-1}} g^{u_i})^\wedge(\omega)}.
\end{aligned}$$

where we make the usual definition $\mathrm{lt}_s f(t) := f(s^{-1}t)$. Therefore, for all $i \geq 0$, we can compute

$$\begin{aligned}
\int_{\widehat{S}} \overline{\hat{g}(\omega)} \hat{f}(\omega) \omega(s_i) d\hat{\beta}^{u_i}(\omega) &= \int_{\widehat{S}_{u_i}} \overline{\widehat{g^{u_i}}(\omega)} \widehat{f^{u_i}}(\omega) \omega(s_i) d\hat{\beta}^{u_i}(\omega) \\
&= \int_{\widehat{S}_{u_i}} \overline{(\mathrm{lt}_{s_i^{-1}} g^{u_i})^\wedge} \widehat{f^{u_i}} d\hat{\beta}^{u_i} = \int_{S_{u_i}} \overline{\mathrm{lt}_{s_i^{-1}} g^{u_i}} f^{u_i} d\beta^{u_i}.
\end{aligned}$$

Using (2.14) it follows that for all $g \in C_c(S)$

$$\int_{S_{u_i}} \overline{\mathrm{lt}_{s_i^{-1}} g^{u_i}} f^{u_i} d\beta^{u_i} \to \int_{S_{u_0}} \overline{\mathrm{lt}_{s_0^{-1}} g^{u_0}} f^{u_0} d\beta^{u_0}. \tag{2.16}$$

Now we are finally ready to attack the convergence of the s_i. Choose an open neighborhood O of s_0. Using the continuity of multiplication we can find relatively compact open neighborhoods V and W in S such that $u_0 \in V$, $s_0 \in W$, and $VW \subset O$. Furthermore, by intersecting V and V^{-1} we can assume that $V^{-1} = V$. Construct

f for V as in the beginning of the proof. Now choose $g \in C_c(S)$ so that $0 \leq g \leq 1$, $g(s_0) = 1$, and g is zero off W. Then $\overline{g} = g$ so that by (2.16) we have

$$\int_{S_{u_i}} g(s_i t) f(t) d\beta^{u_i}(t) \to \int_{S_{u_0}} g(s_0 t) f(t) d\beta^{u_0}(t). \tag{2.17}$$

Given $i \geq 0$, $\int_{S_{u_i}} g(s_i t) f(t) d\beta^{u_i}(t) = 0$ unless there exists t such that $s_i t \in \operatorname{supp} g \subset W$ and $t \in \operatorname{supp} f \subset V$. This implies that the integral is zero unless $s_i \in WV^{-1} = WV \subset O$. Furthermore, both $g(s_0 u_0)$ and $f(u_0)$ are nonzero by construction and, since both are positive continuous functions, this implies $\int_{S_{u_0}} g(s_0 t) f(t) d\beta^{u_0}(t) \neq 0$. It follows from (2.17) that eventually $\int_{S_{u_i}} g(s_i t) f(t) d\beta^{u_i}(t) \neq 0$ so that, eventually, $s_i \in O$. Of course, it follows that $s_i \to s_0$ and we are done. □

2.3 Open Mapping Counterexample

As we noted in Section 2.2, given a second countable continuously varying abelian group bundle S it is easy to see that the natural map from S to its double dual is a continuous, bijective, groupoid homomorphism. Furthermore, in the second countable locally compact Hausdorff group case this map would automatically have a continuous inverse. If this kind of "open mapping theorem" were true for second countable group bundles then Theorem 2.52 would be trivial. In this section we will exhibit an example which shows that not every continuous bijective groupoid homomorphism between second countable, abelian, continuously varying, group bundles is necessarily a homeomorphism. Specifically we will prove the following

Theorem 2.53. *There exists second countable, locally compact Hausdorff, abelian group bundles S and T and a map $\phi : T \to S$ such that ϕ is a continuous, bijective, groupoid homomorphism which is not a homeomorphism.*

Finding an example of such a homomorphism is especially tricky. Fibrewise such a map is a continuous bijective group homomorphism of second countable groups, and as such must be a homeomorphism when restricted to the fibres. Of course, we have to start by defining S and T.

Remark 2.54. In this section it will be convenient to define

$$\mathbb{Z}_{2n+1} = \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$$

for all $n \in \mathbb{N}$. Furthermore, when appropriate we will still give $\mathbb{Z}_{2n+1}$ the group operation of addition modulo $2n+1$. For the remainder of this section we are also going to define $x_n = 1/n$ for $n > 0$ and $x_0 = 0$ and will let $X = \{x_n\}_{n=0}^{\infty}$.

The following technical lemma will see a lot of use in this section.

Lemma 2.55. *Suppose $x_{n_i} \to x_N$ converges in X. Then we can pass to a subsequence, relabel, and assume either*

(a) *$n_i = N$ for all i, or*

(b) *$N = 0$, $n_i \to \infty$, and $n_i \neq 0$ for all i.*

Proof. There are two cases to consider. First, if $N \neq 0$ then, because $x_{n_i} \to x_N$ and x_N is open as a point in X, eventually $x_{n_i} = x_N$. Therefore, $n_i = N$ eventually, and by passing to a subsequence we can assume that $n_i = N$ for all i. On the other hand, suppose $N = 0$. If $n_i = 0$ infinitely often then we can pass to a subsequence and assume that $n_i = N = 0$ for all i. If it is not true that $n_i = 0$ infinitely often then, because $x_{n_i} \to x_0$, we know that n_i does not equal any $N \in \mathbb{N}$ infinitely often. However, this implies that $n_i \to \infty$. We could then pass to a subsequence again and assume that $n_i \neq 0$ for all i. □

Next we define the total space of one of our bundles and show that it is topologically well behaved.

Proposition 2.56. *Let*

$$T = \{(m, x_n) \in \mathbb{Z} \times X : m \in \mathbb{Z}_{6n+3} \text{ if } n > 0 \text{ and } m \in \mathbb{Z} \text{ otherwise}\}.$$

Then T is closed in $\mathbb{Z} \times X$, and hence second countable locally compact Hausdorff. Furthermore the map $p_T : T \to X$ defined by projection onto the second factor is continuous and open. This makes T into a bundle over X with fibres denoted T_n.

Proof. Suppose $(m_i, x_{n_i}) \to (m, x_N)$ in $\mathbb{Z} \times X$. We would like to show that $(m, x_N) \in T$. Apply Lemma 2.55 and pass to a subsequence. We know that there are two cases to consider. First, suppose that $n_i = N$ for all i. Then for all i we have either $N \neq 0$ and $m = m_i \in \mathbb{Z}_{6n_i+3} = \mathbb{Z}_{6N+3}$ or $N = 0$ and $m = m_i \in \mathbb{Z}$. It follows that $(m, x_N) \in T$. In the second case $N = 0$, but then $(m, x_0) \in T$ automatically. Thus T is closed and since it is a closed subset of a second countable locally compact Hausdorff space it is also second countable locally compact Hausdorff.

Let $p_T : T \to X$, denoted by p when convenient, be given by $p(m, x) = x$ for $(m, x) \in T$. Since p is the restriction of a continuous map it is continuous. We will show p is open. Suppose $x_{n_i} \to x_N$ and that $(m, x_N) \in T$. We will show that we can pass to a subsequence, relabel, and find m_i such that $(m_i, x_{n_i}) \to (m, x_N)$ and $(m_i, x_{n_i}) \in T$ for all i. By applying Lemma 2.55, and possibly passing to a subsequence and relabeling, we can either assume that $n_i = N$ eventually or that $N = 0$, $n_i \to \infty$ and $n_i \neq 0$ for all i. Suppose the former is true. Then either $N \neq 0$

and $m \in \mathbb{Z}_{6n_i+3} = \mathbb{Z}_{6N+3}$ or $N = 0$ and $m \in \mathbb{Z}$ for all i so that $(m, x_{n_i}) \in T$ for all i and clearly $(m, x_{n_i}) \to (m, x_N)$. Suppose the latter is true. Then eventually $|m| \leq 3n_i + 1$ and by passing to a subsequence we can assume that this is true for all i. Hence $m \in \mathbb{Z}_{6n_i+1}$ and $(m, x_{n_i}) \in T$ for all i, and clearly $(m, x_{n_i}) \to (m, x_0)$. It now follows from Proposition 1.25 that p is open. □

Next, we add the groupoid structure on T.

Proposition 2.57. *If we endow T_n with the operations of addition modulo $6n + 3$ for $n > 0$ and T_0 with the usual integer addition, then T is a second countable, continuously varying, abelian, locally compact Hausdorff group bundle and the bundle map for T can be identified with p.*

Proof. It follows from Proposition 2.56 that T has the appropriate topological conditions. Furthermore, it is easy to see that, using the operations defined in the statement of the proposition, T is an abelian group bundle with unit space X and that the bundle map is exactly p. After all, algebraically T is just the disjoint union of the groups T_n.

All that is left is to show that the operations on T are continuous. Suppose $(m_i, x_{n_i}) \to (x_N)$ and $(m'_i, x_{n_i}) \to (m', x_N)$ in T. We would like to show that

$$(m_i, x_{n_i}) + (m'_i, x_{n_i}) \to (m, x_N) + (m', x_N), \quad \text{and}$$
$$-(m_i, x_{n_i}) \to -(m, x_N).$$

It will suffice to show that for each subsequence, a sub-subsequence converges in the above fashion. So pass to a subsequence. Using Lemma 2.55 we can pass to another subsequence and either assume that $n_i = N$ for all i or that $N = 0$ and $n_i \to \infty$. Suppose the former is true. Then, eventually $m_i = m$ and $m'_i = m'$ so that eventually $(m_i, x_{n_i}) + (m'_i, x_{n_i}) = (m, x_N) + (m', x_N)$. Similarly in this case $-(m_i, x_{n_i}) = -(m, x_N)$ and at this point convergence is clear. Now suppose we are in the second case so that $n_i \to \infty$, $n_i \neq 0$ for all i and $N = 0$. As before we have $m_i = m$ and $m'_i = m'$ eventually so that, for large i, $|m_i + m'_i| = |m + m'| \leq n_i$. However, when this is true $m + m' \mod 6n_i + 3 = m + m'$. It follows that for large enough i

$$(m_i, x_{n_i}) + (m'_i, x_{n_i}) = (m + m', x_{n_i})$$

and it is clear that $(m + m', x_{n_i}) \to (m + m', x_0) = (m, x_0) + (m', x_0)$. Since $-(m_i, x_{n_i}) = (-m_i, x_{n_i})$ for all i it is easier to see that, assuming i is large enough,

$$-(m_i, x_{n_i}) = (-m, x_{n_i}) \to (-m, x_0) = -(m, x_0).$$

It follows that the operations on T are continuous and that T is a topological groupoid. However, we can now conclude that T is continuously varying, since the bundle map

p is open. $\square$

Remark 2.58. Since T is clearly r-discrete the Haar system on T is given by $\lambda^{x_n} = \mu \times \delta_{x_n}$ where μ is counting measure and δ_{x_n} is the Dirac delta measure at x_n.

Next we define the total space of our other bundle and show that it is well behaved topologically.

Proposition 2.59. *Let $A_n = \{-1/n, 0, 1/n\}$ and $S_n = A_n \times \mathbb{Z}_{2n+1} \times \{x_n\}$ for $n > 1$. Let $A_0 = \{0\}$ and $S_0 = A_0 \times \mathbb{Z} \times \{x_0\}$. Define*

$$S = \bigcup_{n=0}^{\infty} S_n$$

and give S the relative topology as a subset of $\mathbb{R} \times \mathbb{Z} \times X$. Then S is closed and is therefore a second countable locally compact Hausdorff space. Furthermore, the map $p_S : S \to X$ defined by projection onto X is continuous and open.

Proof. Suppose $(a_i, m_i, x_{n_i}) \to (a, m, x_N)$ in S. Using Lemma 2.55, pass to a subsequence, relabel, and assume that either $n_i = N$ for all i or that $n_i \to \infty$, $n_i \neq 0$ for all i and $N = 0$. Consider the first case. Observe that eventually either $N \neq 0$ and $m_i = m \in \mathbb{Z}_{2n_i+1} = \mathbb{Z}_{2N+1}$ or $N = 0$ and $m_i = m \in \mathbb{Z}$. Furthermore, we also have, for large i, $a_i \in A_{n_i} = A_N$. Since A_N is a closed set (it consists of at most three distinct points), $a = \lim a_i \in A_N$. Thus $(a, m, x_N) \in S_N \subset S$. On the other hand suppose that $n_i \to \infty$ and $n_i \neq 0$ for all i. Since $-1/n_i \leq a_i \leq 1/n_i$ for all i we can conclude that $0 = \lim a_i = a$. Hence $(a, m, x_N) \in S_0 \subset S$. It follows that S is closed.

Let $p_S : S \to X$, often denoted by p, be defined by $p_S(a, m, x) = x$. Since p is the restriction to S of a continuous map it must be continuous. We will show that it is open. Suppose $x_{n_i} \to x_N$ and that $(a, m, x_N) \in S$. Then, using Lemma 2.55, we pass to a subsequence, relabel, and assume that either $n_i = N$ for all i or that $n_i \to \infty$, $n_i \neq 0$ for all i, and $N = 0$. Suppose the former is true. Then either $N \neq 0$, $m \in \mathbb{Z}_{2N+1} = \mathbb{Z}_{2n_i+1}$, and $a \in A_N = A_{n_i}$ for all i or $N = 0$, $m \in \mathbb{Z}$, and $a = 0$. It follows that $(a, m, x_{n_i}) \in S_{n_i}$ for all i and we clearly have $(a, m, x_{n_i}) \to (a, m, x_N)$. On the other hand suppose the latter case is true. Since $N = 0$ we know $a = 0$. Furthermore $|m| < n_i$ eventually, so we might as well pass to a subsequence and assume that this is always true. It follows that $m \in Z_{2n_i+1}$ for all i and that $(0, m, x_{n_i}) \in S_{n_i}$ for all i. Furthermore it is clear that $(a, m, x_{n_i}) \to (a, m, x_N)$. It follows that p is open. $\square$

Lemma 2.60. *Given $m \in \mathbb{Z}$ and $n > 0$ there exists a unique $d \in \mathbb{Z}$ and $r \in \mathbb{Z}$ such that $m = d(2n+1) + r$ and $-n \leq r \leq n$. Furthermore if $|m| \leq 6n + 3$ then $-1 \leq d \leq 1$.*

Proof. This is a straightforward modification of the division algorithm where we divide by $2n+1$ and allow the remainder to take on values between $-n$ and n. Furthermore if $|d| > 1$ then, as long as $-n \leq r \leq n$, we have $|m| > 6n+3$. $\square$

Lemma 2.61. *For $d \in \mathbb{Z}_3$ let $a^n(d) = d/n$ for all $n > 0$ and let $a^0(d) = 0$. Then every element of A_n is of the form $a^n(d)$ for $d \in \mathbb{Z}_3$. Furthermore, given $n > 0$ we can define a bijective map $\phi : T_n \to S_n$ such that $\phi_n(m, x_n) = (a^n(d), r, x_n)$ where d and r are as in Lemma 2.60. Finally, the map $\phi_0 : T_0 \to S_0$ such that $\phi(m, x_0) = (0, m, x_0)$ is a bijection.*

Proof. First, it is clear from the definition of A_n that every element is of the form $a^n(d)$ for $d = -1, 0, 1$. Let $n > 0$. Given $(m, x_n) \in S_n$ we know $|m| \leq 6n+3$ by definition. Let $d, r \in \mathbb{Z}$ be as in Lemma 2.60. Then $r \in \mathbb{Z}_{2n+1}$ and, since $|d| \leq 1$, we know $a^n(d) \in A_n$. Hence $\phi_n(m, x_n) = (a^n(d), r, x_n) \in T_n$ and ϕ_n is well defined. Furthermore, if $\phi_n(m, x_n) = (a^n(d), r, x_n) = \phi(m', x_n)$ then $m = d(2n+1) + r = m'$ so that $(m, x_n) = (m', x_n)$. Next, given $(a, r, x_n) \in S_n$ choose $d \in \mathbb{Z}_3$ so that $a = a^n(d)$. Then $m = d(2n+1) + r \in \mathbb{Z}_{6n+3}$ and, by the uniqueness of the factorization, $\phi(m, x_n) = (a, r, x_n)$. It follows that ϕ_n is a bijection. Finally, it is clear that ϕ_0 is a bijection. $\square$

Next, we use the above maps to define the group structure on the fibres.

Proposition 2.62. *Endow S_0 with the operations of integer addition and negation in the second factor. For $n > 0$ define operations on S_n via*

$$-(a, r, x_n) = (-a, -r, x_n) \tag{2.18}$$

$$(a^n(d), r, x_n) + (a^n(d'), r', x_n) = \tag{2.19}$$

$$\begin{cases} (a^n(d+d'+1 \mod 3), r+r'-(2n+1), x_n) & r+r' > n \\ (a^n(d+d' \mod 3), r+r', x_n) & -n \leq r+r' \leq n \\ (a^n(d+d'-1 \mod 3), r+r'+(2n+1), x_n) & r+r' < -n. \end{cases}$$

With these operations S_n is an abelian group for all $n \geq 0$ and ϕ_n is a group isomorphism for all $n \geq 0$.

Proof. First observe that the topologies on S_n and T_n are discrete for all n so that ϕ_n is a homeomorphism for all n. All we need to do is show that each ϕ_n preserves the operations. Then the operations on S_n will automatically make S_n into an abelian group and ϕ_n will be an isomorphism. This is trivial for ϕ_0.

Let $n > 0$. Verifying that ϕ_n is a homomorphism is straightforward but tedious. For example, if $(a^n(d), r, x_n), (a^n(d'), r', x_n) \in S_n$ such that $r + r' > n$ then let $m =$

$d(2n+1)+r$ and $m' = d'(2n+1)+r'$. Observe that $2n \geq r+r' > n$ so that $-n \leq r+r'-(2n+1) \leq n$. Next we compute

$$m+m' = (d+d')(2n+1)+r+r' = (d+d'+1)(2n+1)+(r+r'-(2n+1)).$$

Suppose $d' = d = 1$. Then $m+m' = 3(2n+1)+(r+r'-(2n+1)) > 3n+1$ since $r+r'-(2n+1) \geq -n$. It follows that, using our version of modulo addition on $\mathbb{Z}_{6n+3}$,

$$\begin{aligned} m+m' \mod 6n+3 &= m+m'-(6n+3) \\ &= (d+d'+1-3)(2n+1)+(r+r'-(2n+1)) \\ &= (d+d+1 \mod 3)(2n+1)+(r+r'-(2n+1)). \end{aligned}$$

The cases for the other possible values of d and d' are similar and in general we have

$$m+m' \mod 6n+3 = (d+d'+1 \mod 3)(2n+1)+(r+r'-(2n+1)).$$

Now, observe that in T_n

$$(m, x_n)+(m', x_n) = (m+m' \mod 6n+3, x_n).$$

By our construction $-n \leq r+r'-(2n+1) \leq n$ and $-1 \leq d+d'+1 \mod 3 \leq 1$ so that, by the definition of ϕ_n and (2.19),

$$\begin{aligned} \phi_n(m+m' \mod 6n+3, x_n) &= (a^n(d+d'+1 \mod 3), r+r'-(2n+1), x_n) \\ &= (a^n(d), r, x_n)+(a^n(d'), r', x_n). \end{aligned}$$

Of course, we have only shown that ϕ_n preserves addition when $r+r' > n$. However, the computations for the other cases are analogous and it is straightforward, but tedious, to see that ϕ_n respects the multiplication operations on S_n and T_n.

Proving that ϕ_n preserves inversion is much easier. Suppose $(m, x_n) \in T_n$. Then if $m = d(2n+1)+r$ is the decomposition given by Lemma 2.60 we have $-m = -d(2n+1)-r$. Since $-a^n(d) = a^n(-d)$ we have

$$\phi_n(-m, x_n) = (a^n(-d), -r, x_n) = -(a^n(d), r, x_n).$$

Thus ϕ_n respects the inverse operation. Since ϕ is a bijection which respects the operations on S_n and T_n the fact that T_n is an abelian group implies that S_n is also an abelian group and that ϕ_n is an isomorphism. □

We have been stepping around it for the last few propositions, but it is time to put everything together and show that we can form a group bundle out of the S_n.

Proposition 2.63. *With the operations on S_n defined in Proposition 2.62 S is an*

abelian, continuously varying, second countable, locally compact Hausdorff group bundle with bundle map p_S.

Proof. First, it follows from Proposition 2.59 that S is second countable, locally compact Hausdorff. It's easy to see from the way we defined S as the disjoint union of groups over X that S is a group bundle with unit space X and bundle map p_S. Furthermore, each S_n is abelian so that S is abelian. All that is left is to show that the operations are continuous. Suppose $(a^{n_i}(d_i), r_i, x_{n_i}) \to (a^N(d), r, x_N)$ and $(a^{n_i}(d_i'), r_i', x_{n_i}) \to (a^N(d'), r', x_N)$ in S. We would like to show that

$$(a^{n_i}(d_i), r_i, x_{n_i}) + (a^{n_i}(d_i'), r_i', x_{n_i}) \to (a^N(d), r, x_N) + (a^N(d'), r', x_N), \quad \text{and}$$
$$-(a^{n_i}(d_i), r_i, x_{n_i}) \to -(a^N(d), r, x_N).$$

It will suffice to show that for every subsequence, we can pass to a sub-subsequence and obtain the required convergence. So, pass to a subsequence and use Lemma 2.55 to pass to another subsequence and assume that either $n_i = N$ for all i or that $n_i \to \infty$, $n_i \neq 0$ for all i, and $N = 0$. Consider the first case. Eventually $r_i = r$ and $r_i' = r'$. Furthermore, since A_N is a discrete space containing $\{a^{n_i}(d_i)\}$ and $\{a^{n_i}(d_i')\}$ we also eventually have $a^{n_i}(d_i) = a^N(d)$ and $a^{n_i}(d_i') = a^N(d')$. Hence, eventually, $d_i = d$ and $d_i' = d'$. However, this implies that for very large i

$$(a^{n_i}(d_i), r_i, x_{n_i}) + (a^{n_i}(d_i'), r_i', x_{n_i}) = (a^N(d), r, x_N) + (a^N(d'), r', x_N), \quad \text{and}$$
$$-(a^{n_i}(d_i), r_i, x_{n_i}) = -(a^N(d), r, x_N)$$

and at this point convergence is clear.

Next, consider the second case so that $n_i \to \infty$ and $n_i \neq 0$ for all i. Given any arbitrary sequence $\{c_i\} \subset \mathbb{Z}_3$ we know that $-1/n_i \leq a^{n_i}(c_i) \leq 1/n_i$ and this implies that $a^{n_i}(c_i) \to 0$. Since $r_i = r$ eventually it follows from (2.18) that

$$-(a^{n_i}(d_i), r_i, x_{n_i}) = (a^{n_i}(-d_i), -r_i, x_{n_i}) \to (0, -r, x_0) = -(a^0(d), r, x_0).$$

Next, eventually $r_i = r$ and $r_i' = r'$ and $|r + r'| < n_i$ so that we can pass to a subsequence and assume this always holds. Furthermore, let $c_i = d_i + d_i' \mod 3$ for all i. Then, by (2.19), we have

$$(a^{n_i}(d_i), r_i, x_{n_i}) + (a^{n_i}(d_i'), r_i', x_{n_i}) = (a^{n_i}(c_i), r + r', x_{n_i}).$$

As before, whatever the c_i, we know that $a^{n_i}(c_i) \to 0$. This implies that

$$(a^{n_i}(c_i), r + r', x_{n_i}) \to (0, r + r', x_0) = (a^0(d), r, x_0) + (a^0(d'), r', x_0).$$

Thus, both of the required sequences converge and it follows that the operations on S

are continuous. This makes S into a topological groupoid, and since we have shown that p_S is open, S is continuously varying. □

Remark 2.64. It is easy enough to see that S is r-discrete so that the Haar system for S is given on S_n by $\mu \times \delta_{x_n}$ where μ is counting measure on $A_n \times \mathbb{Z}_{2n+1}$ (or $A_0 \times \mathbb{Z}$) and δ_{x_n} is the Dirac delta measure at x_n.

We now prove the main result of the section.

Proof of Theorem 2.53. Let S be as in Proposition 2.59 and T be as in Proposition 2.56. Define $\phi : T \to S$ such that $\phi(m, x_n) = \phi_n(m, x_n)$ for all $(m, x_n) \in T$. Algebraically everything is straightforward. It is clear that we can define such a map on T and that, since each ϕ_n is a group homomorphism, the resulting ϕ will be a groupoid homomorphism. Furthermore, each ϕ_n is bijective so ϕ is bijective. Let us show that it is continuous. Suppose $(m_i, x_{n_i}) \to (m, x_N)$. As before it suffices to show that given a subsequence of $\phi(m_i, x_{n_i})$ we can find a sub-subsequence converging to $\phi(m, x_N)$. So let us pass to a subsequence, and then do so again to assume that $m_i = m$ for all i. Now use Lemma 2.55 to pass to yet another subsequence and assume that either $n_i = N$ for all i or that $n_i \to \infty$, $n_i \neq 0$ for all i, and that $N = 0$. When the former is true (m_i, x_{n_i}) is a constant sequence so of course $\phi(m_i, x_{n_i})$ converges to $\phi(m, x_N)$. Suppose the latter is true. Eventually $|m| \leq n_i$ and when this happens $\phi(m, x_{n_i}) = (0, m, x_{n_i})$. Clearly $(0, m, x_{n_i}) \to (0, m, x_0)$ and therefore ϕ is continuous.

Now consider the sequence $(2n + 1, x_n)$. It is easy to see that $\phi(2n + 1, x_n) = (1/n, 0, x_n)$ so that clearly

$$\phi(2n + 1, x_n) = (1/n, 0, x_n) \to (0, 0, x_0) = \phi(0, x_0).$$

However, the sequence $(2n + 1, x_n)$ does not converge to anything in T. It follows that ϕ is not a homeomorphism. □

Chapter 3

Groupoid Crossed Products

In this chapter we give the definition of a groupoid dynamical system and construct the groupoid crossed product. Unfortunately, many elements of the construction are rather technical and we will need to draw upon a wealth of existing mathematics. In Section 3.1 we give a brief overview of upper-semicontinuous bundles and their relation to $C_0(X)$-algebras. In Section 3.2 we define a groupoid dynamical system and construct the function algebra from which we will build the crossed product. Section 3.3 concerns covariant representations. In order to properly define a covariant representation we will need to deal with both groupoid representations and decompositions of representations of C^*-algebras. After developing these tools we then construct the crossed product in Section 3.4. The most important result in this section is Renault's Disintegration Theorem which will free us from having to deal with covariant representations directly. In particular, the beginner should not be discouraged if they don't understand all of Section 3.3 on their first read through.

3.1 Upper-semicontinuous Bundles

This section is essentially a collection of the important results concerning upper-semicontinuous bundles that we will need for the study of groupoid crossed products. Those readers unfamiliar with $C_0(X)$-algebras and their related bundles are referred to [Wil07, Appendix C]. This reference is self-contained and does a very good job of covering the basics of $C_0(X)$-algebra theory. In fact, for the most part, the definitions and theorems in this section are lifted from [Wil07] and we will cite a number of results from this reference without proof. All of this theory has its roots in [DG83].

Our main concern will be to develop a theory of bundles of C^*-algebras. However, in order to define our induction techniques in Section 6.1 we will need to start with something a little more general.

Definition 3.1. An *upper-semicontinuous Banach bundle* over a locally compact Hausdorff space X is a topological space $\mathcal{A}$ together with a continuous, open surjection $p = p_{\mathcal{A}} : \mathcal{A} \to X$ and complex Banach space structures on each fibre $\mathcal{A}_x := p^{-1}(x)$ satisfying the following axioms.

(a) The map $a \mapsto \|a\|$ is upper-semicontinuous from $\mathcal{A}$ to $\mathbb{R}^+$. (That is, for all $\epsilon > 0$, the set $\{a \in \mathcal{A} : \|a\| \geq \epsilon\}$ is closed.)

(b) If $\mathcal{A} * \mathcal{A} := \{(a, b) \in \mathcal{A} \times \mathcal{A} : p(a) = p(b)\}$, then $(a, b) \mapsto a + b$ is continuous from $\mathcal{A} * \mathcal{A}$ to $\mathcal{A}$.

(c) For each $\lambda \in \mathbb{C}$, $a \mapsto \lambda a$ is continuous from $\mathcal{A}$ to $\mathcal{A}$.

(d) If $\{a_i\}$ is a net in $\mathcal{A}$ such that $p(a_i) \to x$ and such that $\|a_i\| \to 0$, then $a_i \to 0_x$ (where 0_x is the zero element of $\mathcal{A}_x$).

The following proposition is something of a utility belt for dealing with upper-semicontinuous bundles. In particular, the fourth part gives us a handle on the topology of the total space, which can be difficult to deal with.

Proposition 3.2. *Suppose $\mathcal{A}$ is an upper-semicontinuous Banach bundle over X with bundle map p. Then the following statements hold.*

(a) *If $a_i \to 0_x$ in $\mathcal{A}$ then $\|a_i\| \to 0$.*

(b) *For all $x \in X$ the topology of A_x as a subset of $\mathcal{A}$ is exactly its norm topology as a Banach space.*

(c) *The map $(\lambda, a) \mapsto \lambda a$ is continuous from $\mathbb{C} \times \mathcal{A}$ into $\mathcal{A}$.*

(d) *Let $\{a_i\}$ be a net in $\mathcal{A}$ such that $p(a_i) \to p(a)$ for some $a \in \mathcal{A}$. Suppose that for all $\epsilon > 0$ there is a net $\{u_i\}$ in $\mathcal{A}$ and $u \in \mathcal{A}$ such that*

(i) *$u_i \to u$ in $\mathcal{A}$,*

(ii) *$p(u_i) = p(a_i)$ for all i,*

(iii) *$\|a - u\| < \epsilon$, and*

(iv) *eventually $\|a_i - u_i\| < \epsilon$.*

Then $a_i \to a$.

Proof. Part **(a)**: Since the norm is upper-semicontinuous on $\mathcal{A}$ the set $\{a \in \mathcal{A} : \|a\| < \epsilon\}$ is open for all $\epsilon > 0$. Thus we eventually have $\|a_i\| < \epsilon$ for all $\epsilon > 0$ and the result is proved.

Part **(b)**: Suppose that $a_i \to a$ in $\mathcal{A}$ with $p(a_i) = p(a)$ for all i. Then $a_i - a \to 0_{p(a)}$ by the continuity of addition and $\|a_i - a\| \to 0$ by part (a). Conversely, if $\|a_i - a\| \to 0$ then $a_i - a \to 0_{p(a)}$ by the last axiom of Definition 3.1, and $a_i \to a$ by the continuity of addition.

Part **(d)**: Since X is Hausdorff we must have $p(u) = p(a)$ so that condition (iii) makes sense. Pass to a subnet of $\{a_i\}$. It will suffice to show that there is sub-subnet converging to a. Since p is open, we can pass to a subnet, relabel, and find $c_i \in \mathcal{A}_{p(a_i)}$ such that $c_i \to a$. Fix $\epsilon > 0$ and choose u_i as in part (d). Since addition is continuous, $c_i - u_i \to a - u$ in $\mathcal{A}$. Since $\|a - u\| < \epsilon$ by assumption, and since $\{b \in \mathcal{A} : \|b\| < \epsilon\}$ is open, we eventually have $\|c_i - u_i\| < \epsilon$. The triangle inequality then implies that we eventually have $\|a_i - c_i\| < 2\epsilon$. As ϵ was arbitrary, we've shown that $\|a_i - c_i\| \to 0$. Therefore axiom (d) implies that $a_i - c_i \to 0_{p(a)}$. Thus

$$a_i = (a_i - c_i) + c_i \to 0_{p(a)} + a = a.$$

Part **(c)**: Suppose $a_i \to a$ in $\mathcal{A}$ and $\lambda_i \to \lambda$ in $\mathbb{C}$. We will apply part (d) with $u_i = \lambda a_i$ and $u = \lambda a$. It is clear that $p(\lambda_i a_i) = p(a_i) \to p(a) = p(\lambda a)$. Suppose $\epsilon > 0$. By axiom (c) we have $\lambda a_i \to \lambda a$. Conditions (ii) and (iii) are both trivial. For condition (iv), first observe that the set $\{b \in \mathcal{A} : \|b\| < \|a\| + 1\}$ is open by axiom (a). Since this set contains a, it eventually contains a_i. It follows immediately that the set $\{\|a_i\|\}$ is bounded. Hence, because $\lambda_i \to \lambda$, we must have, eventually,

$$\|\lambda_i a_i - \lambda a_i\| = |\lambda_i - \lambda| \|a_i\| < \epsilon$$

and therefore condition (iv) holds. Thus $\lambda_i a_i \to \lambda a$. □

Remark 3.3. What makes the proof of part (c) so complicated is that formulas like $\|\lambda_i a_i - \lambda a\|$ don't make sense because a_i and a could possibly live in different fibres.

Next, we can add structure to an upper-semicontinuous Banach bundle to make it a bundle of C^*-algebras in the obvious way.

Definition 3.4. An *upper-semicontinuous C^*-bundle* is an upper-semicontinuous Banach bundle $p : \mathcal{A} \to X$ such that each fibre is a C^*-algebra and such that the following additional axioms hold.

- **(e)** The map $(a, b) \mapsto ab$ is continuous from $\mathcal{A} * \mathcal{A}$ to $\mathcal{A}$.
- **(f)** The map $a \mapsto a^*$ is continuous from $\mathcal{A}$ to $\mathcal{A}$.

There is also the more restrictive notion of continuous bundles which deserves to be mentioned.

Definition 3.5. An upper-semicontinuous Banach bundle (resp. C^*-bundle) $\mathcal{A}$ is a Banach bundle (resp. C^*-bundle) if the map $a \mapsto \|a\|$ is continuous.

It may seem strange that we are working with upper-semicontinuous bundles as opposed to continuous bundles. However, we will see that, at least in the C^*-algebraic case, upper-semicontinuous bundles are the more natural object.

Definition 3.6. Suppose $\mathcal{A}$ and $\mathcal{B}$ are upper-semicontinuous Banach bundles (resp. C^*-bundles) over X with bundle maps p and q respectively. A continuous map $\phi : \mathcal{A} \to \mathcal{B}$ is called a Banach bundle (resp. C^*-bundle) homomorphism if $q \circ \phi = p$ and for each $x \in X$ the restriction $\phi_x : \mathcal{A}_x \to \mathcal{B}_x$ is a Banach space (resp. C^*-algebra) homomorphism. A Banach bundle (resp. C^*-bundle) *isomorphism* is a bijective, bicontinuous, Banach bundle (resp. C^*-bundle) homomorphism.

Given an upper-semicontinuous bundle the primary object of interest will be the space of sections.

Definition 3.7. Suppose $\mathcal{A}$ is an upper-semicontinuous Banach bundle. Then we will denote the space of sections of the bundle map by $\Gamma(X, \mathcal{A})$. Given $f \in \Gamma(X, \mathcal{A})$ we say that f *vanishes at infinity* if the set $\{x \in X : \|f(x)\| \geq \epsilon\}$ is compact for all $\epsilon \geq 0$. We will denote the subspace of sections which vanish at infinity by $\Gamma_0(X, \mathcal{A})$. Furthermore, we will let $\Gamma_c(X, \mathcal{A})$ be the subspace of sections which have compact support.

We endow $\Gamma(X, \mathcal{A})$ with the operations of pointwise addition and pointwise scalar multiplication. Furthermore we equip $\Gamma_0(X, \mathcal{A})$ with the uniform norm $\|f\|_\infty = \sup_{x\in X} \|f(x)\|$. If $\mathcal{A}$ is an upper-semicontinuous C^*-bundle then we give $\Gamma(X, \mathcal{A})$ the operations of pointwise multiplication and involution. Finally, given $\phi \in C(X)$ and $f \in \Gamma(X, \mathcal{A})$ we define the section $\phi \cdot f$ via $\phi \cdot f(x) := \phi(x)f(x)$ for all $x \in X$.

Remark 3.8. It is not clear at the outset that there are any nontrivial sections in $\Gamma(X, \mathcal{A})$. A bundle $\mathcal{A}$ is said to have *enough sections* if given $x \in X$ and $a \in \mathcal{A}_x$ there exists $f \in \Gamma(X, \mathcal{A})$ such that $f(x) = a$. If we are dealing with a Banach bundle then it is a result of Douady and Soglio-Hérault that there are enough sections [FD88, Appendix C]. Hoffman has noted that the same is true for upper-semicontinuous Banach bundles [Hof74], although the details remain unpublished [Hof77]. We will not need to worry about this because, as we show in Remark 3.28, in all of our examples there will obviously be enough sections.

The point of all this is that the objects in Definition 3.7 have fairly nice algebraic properties and will fill roles analogous to $C(X)$, $C_0(X)$ and $C_c(X)$.

Proposition 3.9. *Suppose $\mathcal{A}$ is an upper-semicontinuous Banach bundle. Then the following hold.*

(a) $\Gamma(X, \mathcal{A})$ *is a vector space with respect to the natural pointwise operations. If $\mathcal{A}$ is a C^*-bundle then* $\Gamma(X, \mathcal{A})$ *is a $*$-algebra.*

(b) *$\Gamma_0(X, \mathcal{A})$ is complete with respect to the uniform norm. Furthermore, $\Gamma_0(X, \mathcal{A})$ is closed under the pointwise operations so that it is a Banach space. If $\mathcal{A}$ is a C^*-bundle then $\Gamma_0(X, \mathcal{A})$ is a C^*-algebra.*

(c) *Given $\phi \in C_0(X)$ and $f \in \Gamma_0(X, \mathcal{A})$ we have $\phi \cdot \gamma \in \Gamma_0(X, \mathcal{A})$ and in particular $\Gamma_0(X, \mathcal{A})$ is a $C_0(X)$-module.*

Proof. The algebraic statements are all straightforward to check. We will content ourselves with showing that $\Gamma_0(X, \mathcal{A})$ is complete. Suppose f_i is a Cauchy sequence in $\Gamma_0(X, \mathcal{A})$. Since each $\mathcal{A}_x$ is complete we can at least define a section $f : X \to \mathcal{A}$ by $f(x) = \lim_i f_i(x)$. Now suppose $\epsilon > 0$ and choose N such that $i, j \geq N$ implies $\|f_i - f_j\|_\infty < \epsilon$. Given $x \in X$ pick $i_x \geq N$ so that $\|f_{i_x}(x) - f(x)\| < \epsilon$. Then for all $x \in X$ and $i \geq N$ we have

$$\|f_i(x) - f(x)\| \leq \|f_i(x) - f_{i_x}(x)\| + \|f_{i_x}(x) - f(x)\| < 2\epsilon.$$

It follows that $\|f_i - f\|_\infty \to 0$. We need to show f is continuous. Suppose $x_i \to x$ and fix $\epsilon > 0$. Choose N so that $\|f_N - f\| < \epsilon$. Since $f_N(x_i) \to f_N(x)$ we can let $a_i = f(x_i)$ and $u_i = f_N(x_i)$ and then part (d) of Proposition 3.2 implies that $f(x_i) \to f(x)$. Next, since $x \mapsto \|f(x)\|$ is the uniform limit of functions which vanish at infinity, it's easy to see that $x \mapsto \|f(x)\|$ vanishes at infinity and therefore $f \in \Gamma_0(X, \mathcal{A})$. ☐

The following proposition gives us another nice tool for dealing with the topology on the total space. It also shows that the topology on $\mathcal{A}$ is determined by its space of sections.

Proposition 3.10. *Let $p : \mathcal{A} \to X$ be an upper-semicontinuous Banach bundle. Suppose that $\{a_i\}$ is a net in $\mathcal{A}$, that $a \in \mathcal{A}$, and that $f \in \Gamma_0(X, \mathcal{A})$ is such that $f(p(a)) = a$. If $p(a_i) \to p(a)$ and if $\|a_i - f(p(a_i))\| \to 0$ then $a_i \to a$.*

Proof. We have $a_i - f(p(a_i)) \to 0$ by axiom (d) of Definition 3.1. However, since f is continuous we also have $f(p(a_i)) \to f(p(a)) = a$. Hence

$$a_i = (a_i - f(p(a_i))) + f(p(a_i)) \to 0_{p(a)} + a = a.$$ ☐

The following proposition is important because it gives a very convenient criterion for a subspace of $\Gamma_0(X, \mathcal{A})$ to be dense. This will be useful because we will often want to use some dense subspace of particularly simple functions. This is proved for upper-semicontinuous C^*-bundles in [Wil07, Proposition C.24] and the extension to Banach bundles is basically the same.

Proposition 3.11. *Suppose $p : \mathcal{A} \to X$ is an upper-semicontinuous Banach bundle over X and Γ is a subspace of $\Gamma_0(X, \mathcal{A})$ such that*

(a) *$f \in \Gamma$ and $\phi \in C_0(X)$ implies $\phi \cdot f \in \Gamma$, and*

(b) *for each $x \in X$ the set $\{f(x) : f \in \Gamma\}$ is dense in $\mathcal{A}_x$.*

Then Γ is dense in $\Gamma_0(X, \mathcal{A})$.

Proof. Fix $f \in \Gamma_0(\mathcal{A})$ and $\epsilon > 0$. We need to find $g \in \Gamma$ such that $\|f - g\|_\infty < \epsilon$. Let K be the compact set $\{x \in X : \|f(x)\| \geq \epsilon/3\}$. Given $x \in K$, there is a $g \in \Gamma$ such that $\|f(x) - g(x)\| < \epsilon/3$. Using upper-semicontinuity, there is a neighborhood U of x such that

$$\|f(y) - g(y)\| < \epsilon/3 \text{ if } y \in U.$$

Since K is compact, there is a cover $U_1, \ldots, U_n$ of K by open sets with compact closure, and $g_i \in \Gamma$ such that

$$\|f(y) - g_i(y)\| < \epsilon/3 \text{ if } y \in U_i.$$

Using [Wil07, Lemma 1.43] we can find a partition of unity $\{\phi_i\}_{i=1}^n \subset C_c(X)$ such that $0 \leq \phi_i(x) \leq 1$ for all $x \in X$, $\operatorname{supp} \phi_i \subset U_i$, if $x \in K$ then $\sum \phi_i(x) = 1$, and if $x \notin K$ then $\sum \phi_i(x) \leq 1$. By assumption, $\sum \phi_i \cdot g_i \in \Gamma$. Now, if $x \in K$ then

$$\begin{aligned}\left\| f(x) - \sum_{i=1}^n \phi_i(x) g_i(x) \right\| &= \left\| \sum_{i=1}^n \phi_i(x)(f(x) - g_i(x)) \right\| \\ &\leq \sum_{i=1}^n \phi_i(x) \|f(x) - g_i(x)\| \\ &\leq \epsilon/3 \leq \epsilon.\end{aligned}$$

But if $x \in U_i \setminus K$, then $\|g_i(x)\| < 2\epsilon/3$. Since $\operatorname{supp} \phi_i \subset U_i$, for any $x \notin K$ we have $\phi_i(x)\|g_i(x)\| \leq \frac{2\epsilon}{3}\phi_i(x)$. Thus if $x \notin K$, we still have

$$\begin{aligned}\left\| f(x) - \sum_i \phi_i(x) g_i(x) \right\| &\leq \|f(x)\| + \sum_i \phi_i(x)\|g_i(x)\| \\ &\leq \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.\end{aligned}$$

Therefore $\sup_{x \in X} \|f(x) - (\sum \phi_i \cdot g_i)(x)\| < \epsilon$ as required. □

3.1.1 $C_0(X)$-algebras

The following objects play the same role for groupoid crossed products that C^*-algebras do for group crossed products. They will eventually explain our preference for upper-semicontinuous bundles over continuous bundles.

Definition 3.12. Suppose that A is a C^*-algebra and that X is a locally compact Hausdorff space. Then A is a $C_0(X)$-*algebra* if there is a homomorphism Φ_A from $C_0(X)$ into the center of the multiplier algebra $ZM(A)$ which is nondegenerate in that the set

$$\Phi_A(C_0(X)) \cdot A := \operatorname{span}\{\Phi_A(f)a : f \in C_0(X), a \in A\}$$

is dense in A.

Remark 3.13. Suppose A is a $C_0(X)$-algebra, $B \subset A$, and $C \subset C_0(X)$. We will use the notation

$$C \cdot B = \Phi_A(C) \cdot B := \operatorname{span}\{\Phi_A(f)a : f \in C, a \in B\}.$$

Our eventual goal is to show that there is a one-to-one correspondence between $C_0(X)$-algebras and upper-semicontinuous C^*-bundles. For starters, what follows next shows how we can view $C_0(X)$-algebras as "fibred" objects.

Proposition 3.14. *Suppose A is a $C_0(X)$-algebra and J is an ideal in $C_0(X)$. Then the closure of $\Phi_A(J) \cdot A$ is an ideal in A.*

Proof. Let I be the closure of $\Phi_A(J) \cdot A$ and observe that I is just the closed linear span of $I_0 = \{\Phi(f)a : f \in J, a \in A\}$. Therefore it will suffice to show that given $a \in A$ and $\Phi(f)b \in I_0$ then $a(\Phi(f)b), (\Phi(f)b)a \in I_0$. However, $\Phi(f)$ is in the center of $M(A)$ so that $a(\Phi(f)b) = \Phi(f)(ab)$ and $(\Phi(f)b)a = \Phi(f)(ba)$. The result follows. $\square$

Definition 3.15. Suppose A is a $C_0(X)$-algebra. Given $x \in X$ let J_x be the ideal of functions in $C_0(X)$ which vanish at x. Then we will denote the ideal $\overline{\Phi_A(J_x) \cdot A}$ by I_x and the quotient A/I_x by $A(x)$. We think of $A(x)$ as the *fibre of* A *over* x and given $a \in A$ we write $a(x)$ for the image of a in $A(X)$. In this way we think of a as a function from X onto the disjoint union $\coprod_{x \in X} A(x)$.

The following are some particularly nice examples of $C_0(X)$-algebras.

Example 3.16. If D is any C^*-algebra and X is a locally compact Hausdorff space then $A = C_0(X, D)$ is a $C_0(X)$-algebra in a natural way:

$$\Phi_A(f)(a)(x) := f(x)a(x)$$

for all $f \in C_0(X)$ and $a \in A$. In this case each fibre $A(x)$ is easily identified with D and the identification of the elements of A with functions on X is the obvious one.

Example 3.17. Suppose that X and Y are locally compact Hausdorff spaces and that $\phi : Y \to X$ is a continuous surjection. Then $C_0(Y)$ becomes a $C_0(X)$-algebra with

respect to the map defined by

$$\Phi_{C_0(Y)}(f)g(y) := f(\phi(y))g(y).$$

The only issue is to see that $\Phi_{C_0(Y)}$ is nondegenerate, but this is easy enough to do using the Stone-Weierstrass theorem. In this example, the fibres $C_0(Y)(x)$ are isomorphic to $C_0(\phi^{-1}(x))$. If $f \in C_0(Y)$ then $f(x)$ is just the restriction of f to $\phi^{-1}(x)$.

Example 3.18. Let $\mathcal{A}$ be an upper-semicontinuous C^*-bundle and $A = \Gamma_0(X, \mathcal{A})$. Then A is a $C_0(X)$-algebra with respect to the map defined by

$$\Phi_A(\phi)f(x) := \phi \cdot f(x) = \phi(x)f(x)$$

for $\phi \in C_0(X)$ and $f \in A$. This is really just [Wil07, Proposition C.23], however everything is fairly straightforward to prove. The only part that could be difficult is the nondegeneracy but this is taken care of by Proposition 3.11. It is also easy enough to show that in this case $A(x) \cong \mathcal{A}_x$ for all $x \in X$. The isomorphism is given on A by evaluation at x so that if $f \in A$ then $f(x)$ as an element of $\mathcal{A}_x$ is identified with $f(x)$ as an element of the quotient $A(x)$.

Next, we define the homomorphisms associated to $C_0(X)$-algebras. In particular we will show that they preserve the "fibering" process.

Definition 3.19. Suppose A and B are $C_0(X)$-algebras. A map $\phi : A \to B$ is called $C_0(X)$-*linear* if $\phi(\Phi_A(f)a) = \Phi_B(f)\phi(a)$ for all $f \in C_0(X)$ and $a \in A$.

Proposition 3.20. *Suppose A and B are $C_0(X)$-algebras and $\phi : A \to B$ is a $C_0(X)$-linear homomorphism. Then, for all $x \in X$, ϕ factors to a homomorphism $\phi_x : A(x) \to B(x)$ such that $\phi_x(a(x)) = \phi(a)(x)$. Furthermore, if ϕ is an isomorphism then each ϕ_x is as well.*

Proof. Given $x \in X$ let J_x be the ideal of functions on $C_0(X)$ vanishing at x. Furthermore, let I_x^A and I_x^B be the ideals in A and B, respectively, such that $A(x) = A/I_x^A$ and $B(x) = B/I_x^B$. We would like to show that $\phi(I_x^A) \subset \phi(I_x^B)$. Since ϕ is a homomorphism and I_x^A is the closure of the set $J_x \cdot A$ it suffices to show that $\phi(f \cdot a) \in I_x^B$ for all $a \in A$ and $f \in J_x$. However $\phi(f \cdot a) = f \cdot \phi(a)$ and the result follows. At this point we can compose ϕ with the quotient map $b \mapsto b(x)$ and this will factor to a homomorphism $\phi_x : A(x) \to B(x)$ defined via $\phi_x(a(x)) = \phi(a)(x)$. Furthermore, if ϕ is an isomorphism then ϕ^{-1} is $C_0(X)$-linear and we can construct $(\phi_x)^{-1}$. However, it is straightforward to check that $(\phi_x)^{-1} = \phi_x^{-1}$ so that in this case each ϕ_x is an isomorphism. $\square$

An essential fact about $C_0(X)$-algebras is that their primitive ideal spaces are fibred over X, and that there is a tight relationship between the action and this fibration.

Remark 3.21. Recall from the Dauns-Hofmann Theorem [RW98, Theorem A.34] that given a C^*-algebra A there is an isomorphism $\Psi : C^b(\operatorname{Prim} A) \to ZM(A)$ given as follows. For $f \in C^b(\operatorname{Prim} A)$ and $P \in \operatorname{Prim} A$ let

$$(\Psi(f)(a))(P) := f(P)a(P) \tag{3.1}$$

where $a(P)$ denotes the image of a in the quotient A/P. Then (3.1) defines a unique element of A which we denote $\Psi(f)(a)$. In this way $\Psi(f)$ defines an element in the center of the multiplier algebra $M(A)$.

Proposition 3.22 ([Wil07, Proposition C.5])**.** *Suppose A is a C^*-algebra and that X is a locally compact Hausdorff space. If there is a continuous map $\sigma_A : \operatorname{Prim} A \to X$ then A is a $C_0(X)$-algebra with*

$$\Phi_A(f)a := \Psi(f \circ \sigma_A)a \tag{3.2}$$

for all $f \in C_0(X)$ and $a \in A$. Conversely, if A is a $C_0(X)$-algebra then there is a continuous map $\sigma_A : \operatorname{Prim} A \to X$ such that (3.2) holds.

In particular, every irreducible representation of A is lifted from a fibre $A(x)$ for some $x \in X$. More precisely, if $\pi \in \widehat{A}$ then the ideal $I_{\sigma_A(\ker \pi)}$ is contained in $\ker \pi$ and π is lifted from an irreducible representation of $A(\sigma_A(\ker \pi))$. In this way we can identify $\widehat{A}$ with the disjoint union $\coprod_{x \in X} A(x)^\wedge$.

Thus, the map σ_A gives us our fibration of $\operatorname{Prim} A$.

Corollary 3.23. *Suppose A is a $C_0(X)$-algebra and $\sigma_A : \operatorname{Prim} A \to X$ is the map given in Proposition 3.22. We can view $\operatorname{Prim} A$ as a bundle over X and the fibre $\sigma_A^{-1}(x)$ can be identified with $\operatorname{Prim} A(x)$ for all $x \in X$.*

Proof. This is nothing more than a restatement of the second part of Proposition 3.22 in terms of primitive ideals. In particular, given $P \in \operatorname{Prim} A$ choose any $\pi \in \widehat{A}$ such that $P = \ker \pi$ and it follows that $I_{\sigma(P)} \subset P$ and that P is lifted from an element of $\operatorname{Prim} A(x)$. □

Proposition 3.22 allows us to present another example of a $C_0(X)$-algebra that will be particularly important in Section 5.4.

Example 3.24. Suppose A is a C^*-algebra with Hausdorff spectrum $\widehat{A}$. Since the spectrum is always locally compact it follows that $\widehat{A}$ is locally compact Hausdorff. It is straightforward to show [RW98, Lemma 5.1] that the map $\pi \mapsto \ker \pi$ induces a

homeomorphism of $\widehat{A}$ onto $\operatorname{Prim} A$. Therefore, if we identify $\operatorname{Prim} A$ with $\widehat{A}$ via this map, then $\sigma_A = \mathrm{id}$ allows us to view A as a $C_0(\widehat{A})$-algebra. Given $f \in C_0(\widehat{A})$ we combine (3.1) and (3.2) to get

$$\Phi_A(f)a(\pi) = f(\pi)a(\pi)$$

where $a(\pi)$ is the image of a in $A/\ker\pi$. From here it is straightforward to identify the fibres as $A(\pi) = A/\ker\pi$. It also follows from [RW98, Lemma 5.1] (and is easy to show directly) that each fibre $A(x)$ is simple and has, up to equivalence, a unique faithful irreducible representation. Moreover, in the separable case each $A(x)$ is elementary

Moving on, the "fibration" of A given by a $C_0(X)$-action is much more rigorous than one might think. The key link between $C_0(X)$-algebras and upper-semicontinuous C^*-bundles is given by the following theorem, which is, more or less, a summary of the results in [Wil07, Appendix C]. It also justifies our preference for upper-semicontinuous bundles since there are many well behaved $C_0(X)$-algebras for which the map $\sigma : \operatorname{Prim} A \to X$ is not open.

Theorem 3.25 ([Wil07, Theorem C.26])**.** *Suppose A is a C^*-algebra. Then the following statements are equivalent.*

- **(a)** *A is a $C_0(X)$-algebra.*
- **(b)** *There is a continuous map $\sigma_A : \operatorname{Prim} A \to X$.*
- **(c)** *There is an upper-semicontinuous C^*-bundle $p : \mathcal{A} \to X$ over X and a $C_0(X)$-linear isomorphism of A onto $\Gamma_0(X, \mathcal{A})$.*

Moreover, $\mathcal{A}$ is a C^*-bundle over X if and only if σ_A is open.

The following corollary is nothing more than a basic rehashing of Theorem 3.25. It is important, however, because it presents the view of $C_0(X)$-algebras and upper-semicontinuous C^*-bundles that we will use from now on.

Corollary 3.26. *Suppose A is a $C_0(X)$-algebra. Then we can endow the disjoint union $\mathcal{A} = \coprod_{x\in X} A(x)$ with a unique topology which makes it into an upper-semicontinuous C^*-bundle such that the map which sends $a \in A$ to the section $x \mapsto a(x)$ is a $C_0(X)$-linear isomorphism of A onto $\Gamma_0(X, \mathcal{A})$. Moreover, every upper-semicontinuous C^*-bundle can be obtained in this fashion.*

Proof. Suppose A is a $C_0(X)$-algebra and $\mathcal{A}$ is defined as above. Let $\mathcal{B}$ be an upper-semicontinuous C^*-bundle such that there is a $C_0(X)$-linear isomorphism $\phi : A \to \Gamma_0(X, \mathcal{B})$. First, we use the canonical action of $C_0(X)$ on $B = \Gamma_0(X, \mathcal{B})$ to view B as

a $C_0(X)$-algebra. Given $x \in X$ let I_x^A be the ideal in A generated by $\Phi_A(J_x) \cdot A$ and I_x^B the ideal in B generated by $\Phi_B(J_x) \cdot B$. For a little while we will use the notation $a + I_x^A := a(x)$ since we don't want to confuse elements of quotients with function evaluation. It follows from Proposition 3.20 that ϕ factors to an isomorphism $\phi_x : A(x) \to B(x)$ which is defined via $\phi_x(a+I_x^A) = \phi(a)+I_x^B$ for all $a+I_x^A \in A(x)$. Next, is not hard to check that $I_x^B := \{f \in \Gamma_0(X, \mathcal{B}) : f(x) = 0\}$ and that $\psi_x(f+I_x^B) = f(x)$ defines an isomorphism of $B(x)$ onto $\mathcal{B}_x$.

Now we can define $\Omega : \mathcal{A} \to \mathcal{B}$ by $\Omega(a) = \psi_{p(a)}(\phi_{p(a)}(a))$ for all $a \in \mathcal{A}$. Once we sort out all of the definitions it is easy to see that Ω is a bijection and that restricted to a fibre $\Omega_x = \psi_x \circ \phi_x$ is a C^*-algebra isomorphism. It follows that we can pull back the topology from $\mathcal{B}$ to $\mathcal{A}$ and, with this topology, $\mathcal{A}$ will be an upper-semicontinuous C^*-bundle. Furthermore, in this situation Ω will be a C^*-bundle isomorphism.

Now we have to see that sections have the right form. It is straightforward to show that the bundle isomorphism Ω induces an isomorphism $\omega : \Gamma_0(X, \mathcal{B}) \to \Gamma_0(X, \mathcal{A})$ by $\omega(f) = \Omega^{-1} \circ f$ for all $f \in \Gamma_0(X, \mathcal{B})$. We can compose ω with ϕ to conclude that A is isomorphic to $\Gamma_0(X, \mathcal{A})$ and, furthermore, we can calculate

$$\begin{aligned}\omega \circ \phi(a)(x) &= \omega(\phi(a))(x) = \Omega^{-1}(\phi(a)(x)) \\ &= \phi_x^{-1} \circ \psi_x^{-1}(\phi(a)(x)) = \phi_x^{-1}(\phi(a) + I_x^B) \\ &= a + I_x^A.\end{aligned}$$

However, reverting back to our former notation, this implies that $\omega \circ \phi(a)$ is exactly the section which sends x to $a(x)$.

Next, let $\Upsilon : A \to \Gamma_0(X, \mathcal{A})$ be given by $\Upsilon(a)(x) = a(x)$ for all $a \in A$ and $x \in X$. Suppose that $\mathcal{A}'$ is equal to $\mathcal{A}$ as a set but has a different topology such that Υ is a $C_0(X)$-isomorphism onto $\Gamma_0(X, \mathcal{A}')$. We will use Υ' to denote this new isomorphism. It follows from Proposition 3.20 that for each $x \in X$ both Υ and Υ' factor to isomorphisms from $A(x)$ to $\mathcal{A}_x$ and $\mathcal{A}'_x$, respectively. Thus, fibrewise $\mathcal{A}$ and $\mathcal{A}'$ have the same norm. Suppose $a_i \to a$ in $\mathcal{A}$ and let $u_i = p(a_i)$ and $u = p(a)$. Choose $b \in A$ such that $b(u) = a$. Observe that $u_i \to u$ and that, by viewing b as a continuous section of $\mathcal{A}$ we have $a_i - b(u_i) \to 0$. It follows from Proposition 3.2 that $\|a_i - b(u_i)\| \to 0$. However, by using Υ' to view b as a section of $\mathcal{A}'$, it follows from Proposition 3.10 that $a_i \to a$ in $\mathcal{A}'$. Since the situation is entirely symmetric this implies that the topology on $\mathcal{A}$ is unique.

Finally, the fact that every upper-semicontinuous C^*-bundle can be obtained in this fashion is an implication of the equivalence in Theorem 3.25. $\square$

Definition 3.27. Given a $C_0(X)$-algebra A we define the upper-semicontinuous C^*-bundle *associated to* A to be $\mathcal{A} = \coprod_{x \in X} A(x)$ with the topology from Corollary 3.26.

Remark 3.28. Observe that if A is a $C_0(X)$-algebra and $\mathcal{A}$ is the upper-semicontinuous bundle associated to A then $\mathcal{A}$ has enough sections. Indeed, if $a \in A(x)$ then we can view $A(x)$ as a quotient of A to find $b \in A$ such that $b(x) = a$. However, we can also view b as a section in $\Gamma_0(X, \mathcal{A})$ which takes on the value a at x.

Remark 3.29. We will need to make sure we don't confuse the $C_0(X)$-algebra A with its associated bundle $\mathcal{A}$. One reason we must do this is because the topology on $\mathcal{A}$ is not at all straightforward and we will need to be extra careful in dealing with it. For instance, $\mathcal{A}$ may not even be Hausdorff [Wil07, Example C.27]. (Although it turns out that $\mathcal{A}$ has to be Hausdorff if it is a continuous bundle.)

This duality between upper-semicontinuous bundles and $C_0(X)$-algebras allows us to construct a similar duality between the homomorphisms of these two categories.

Proposition 3.30. *Suppose A and B are $C_0(X)$-algebras and $\mathcal{A}$ and $\mathcal{B}$ are the associated upper-semicontinuous bundles. Then a $C_0(X)$-linear homomorphism $\phi : A \to B$ induces a C^*-bundle homomorphism $\hat{\phi} : \mathcal{A} \to \mathcal{B}$ via $\hat{\phi}(a(x)) = \phi(a)(x)$ for all $a(x) \in \mathcal{A}$.*

Conversely, a C^*-bundle homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ induces a $C_0(X)$-linear homomorphism $\check{\phi} : A \to B$ where $\check{\phi}(a)$ is uniquely determined by the relation $\check{\phi}(a)(x) = \phi(a(x))$ for all $x \in X$.

Proof. This is really a matter of sorting out definitions. Given a $C_0(X)$-linear map $\phi : A \to B$ it follows from Proposition 3.20 that, for each $x \in X$, there is a well defined homomorphism $\phi_x : A(x) \to B(x)$ defined by $\phi_x(a(x)) = \phi_x(a)(x)$. We can glue each of these homomorphisms together to get the map $\hat{\phi} : \mathcal{A} \to \mathcal{B}$. It is clear that $\hat{\phi}$ preserves fibres and that restricted to fibres $\hat{\phi}$ is a homomorphism. All we need to do is show that $\hat{\phi}$ is continuous. Suppose $b_i \to b$ in $\mathcal{A}$ and let $x_i = p(b_i)$ and $x = p(b)$. Lift b from the quotient $A(x)$ to find $a \in A$ such that $a(x) = b$. First, observe that because p is continuous $x_i \to x$. Next, observe that $b_i - a(x_i) \to 0_x$ so that by Proposition 3.2 $\|b_i - a(x_i)\| \to 0$. Since ϕ_x is contractive for all x, we have $\|\phi_{x_i}(b_i) - \phi_{x_i}(a(x_i))\| \leq \|b_i - a(x_i)\|$ so that, using the definition of ϕ_{x_i},

$$\|\phi_{x_i}(b_i) - \phi(a)(x_i)\| \to 0.$$

However, $\phi(a)$ is a section of $\mathcal{B}$ such that $\phi(a)(x) = \phi_x(a(x)) = \phi_x(b)$ so that it follows from Proposition 3.10 that $\phi_{x_i}(b_i) \to \phi_x(b)$. For the reverse direction, identify A and B as the section algebras of $\mathcal{A}$ and $\mathcal{B}$ respectively and define $\check{\phi} : A \to B$ by $\check{\phi}(a) = \phi \circ a$. The result follows without too much difficulty. □

Corollary 3.31. *Suppose A and B are $C_0(X)$-algebras and $\mathcal{A}$ and $\mathcal{B}$ are the associated upper-semicontinuous bundles. If $\phi : A \to B$ is a $C_0(X)$-linear isomorphism then $\hat{\phi}$ is a C^*-bundle isomorphism. Conversely, if $\phi : \mathcal{A} \to \mathcal{B}$ is a C^*-bundle isomorphism then $\check{\phi}$ is a $C_0(X)$-linear isomorphism.*

Proof. For the first direction, use Proposition 3.30 on both ϕ and ϕ^{-1}. Then use the characterization of $\hat{\phi}$ and $\hat{\phi}^{-1}$ to show that these maps are inverses. The other direction is exactly the same. $\square$

Remark 3.32. It follows from Corollary 3.31 that two $C_0(X)$-algebras are $C_0(X)$-isomorphic if and only if their associated bundles $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Thus, citing Theorem 3.25, the map sending A to its associated bundle is a bijection between isomorphism classes of $C_0(X)$-algebras and upper-semicontinuous C^*-bundles.

3.1.2 Pull Back Bundles

The last bit of $C_0(X)$-algebra theory that we need is the notion of a pull back.

Definition 3.33. Suppose X and Y are locally compact Hausdorff spaces, $\mathcal{A}$ is an upper-semicontinuous Banach bundle over X, and that $\tau : Y \to X$ is continuous. The *pull back* of $\mathcal{A}$ is defined to be the set

$$\tau^*\mathcal{A} = \{(y, a) \in Y \times \mathcal{A} : \tau(y) = p(a)\}.$$

In this case $\tau^*\mathcal{A}$ is equipped with the relative topology and the bundle map $q : \tau^*\mathcal{A} \to Y$ defined by $q(y, a) = y$.

Of course, we made this definition with every intention of proving the following

Proposition 3.34. *Suppose X and Y are locally compact Hausdorff spaces, $\mathcal{A}$ is an upper-semicontinuous Banach bundle, and $\tau : Y \to X$ is continuous. Then the pull back $\tau^*\mathcal{A}$ is an upper-semicontinuous Banach bundle. What's more, $\tau^*\mathcal{A}$ is an upper-semicontinuous C^*-bundle if $\mathcal{A}$ is, and if $\mathcal{A}$ is a continuous bundle then $\tau^*\mathcal{A}$ is as well.*

Proof. First, observe that $\tau^*\mathcal{A}_y$ can be easily identified with $\mathcal{A}_{\tau(y)}$ so that we can give $\tau^*\mathcal{A}_y$ whatever structure $\mathcal{A}_{\tau(y)}$ has. Next, note that the bundle map $q : \tau^*\mathcal{A} \to Y$ is continuous since it's the restriction of a continuous map. Let us show that it is open. Suppose $y_i \to y$ in Y and $a \in \mathcal{A}_{\tau(y)}$. Then $\tau(y_i) \to \tau(y)$ and we can use the fact that the bundle map for $\mathcal{A}$ is open to pass to a subnet, relabel, and find $a_i \in \mathcal{A}_{\tau(y_i)}$ such that $a_i \to a$. It follows that $(y_i, a_i) \to (y, a)$ so that q is open.

All that is left is to verify the various bundle axioms. The axioms concerning the continuity of the operations are straightforward, as is axiom (d). We will content ourselves with showing that axiom (a) holds. Suppose $\epsilon > 0$. We would like to show that the set $C = \{(y, a) \in \tau^*\mathcal{A} : \|a\| \geq \epsilon\}$ is closed. Suppose $\{(y_i, a_i)\}$ is a net in C and that $(y_i, a_i) \to (y, a)$. Since $\mathcal{A}$ is an upper-semicontinuous bundle $\|a\| \geq \epsilon$ and we are done. Finally, if $a \mapsto \|a\|$ is continuous then clearly its composition with $(y, a) \mapsto a$ is continuous. $\square$

Proposition 3.35. *Suppose X and Y are locally compact Hausdorff spaces, $\mathcal{A}$ is an upper-semicontinuous Banach bundle, and $\tau : Y \to X$ is continuous. Then $f \in \Gamma(Y, \tau^*\mathcal{A})$ if and only if there exists a continuous function $\tilde{f} : Y \to \mathcal{A}$ such that $p(\tilde{f}(y)) = \tau(y)$ and $f(y) = (y, \tilde{f}(y))$ for all $y \in Y$. Furthermore, $\tilde{f}$ is compactly supported if and only if f is as well.*

Proof. Given $f \in \Gamma(Y, \tau^*\mathcal{A})$ we define $\tilde{f}$ to be the composition of f with the projection from $\tau^*\mathcal{A}$ onto $\mathcal{A}$. Given a continuous $\tilde{f} : Y \to \mathcal{A}$ such that $p(\tilde{f}(y)) = \tau(y)$ we define f by $f(y) = (y, \tilde{f}(y))$. Given f and $\tilde{f}$ as in the statement of the proposition it is clear that $\|f(y)\| = \|\tilde{f}(y)\|$. It follows immediately that f is compactly supported if and only if $\tilde{f}$ is as well. □

Remark 3.36. We will often times denote the element $(y, a) \in \tau^*\mathcal{A}$ by just a and will usually not distinguish between the maps f and $\tilde{f}$.

Definition 3.37. Let X and Y be locally compact Hausdorff spaces, A be a $C_0(X)$-algebra, $\mathcal{A}$ its associated upper-semicontinuous C^*-bundle, and $\tau : X \to Y$ a continuous map. We define the *pull back* of A to be $\tau^*A := \Gamma_0(Y, \tau^*\mathcal{A})$.

Proposition 3.38. *Let X and Y be locally compact Hausdorff spaces, A be a $C_0(X)$-algebra, and $\tau : Y \to X$ a continuous map. Then there is a natural identification of $\tau^*A(y)$ with $A(\tau(y))$ for all $y \in Y$.*

Proof. This is really just working out the definitions. Let $\mathcal{A}$ be the bundle associated to A so that $\tau^*A = \Gamma_0(Y, \tau^*\mathcal{A})$. Then, as we have seen in Example 3.18, $\tau^*A(y) = \tau^*\mathcal{A}_y$. It follows, almost by definition, that $\tau^*\mathcal{A}_y = \mathcal{A}_{\tau(y)} = A(\tau(y))$ and we are done. □

Remark 3.39. When τ is a surjection, τ^*A is usually defined to be the balanced tensor product $C_0(Y) \otimes_{C_0(X)} A$ where we view $C_0(Y)$ as a $C_0(X)$-algebra as in Example 3.17. We will show that this is equivalent to our definition in Section 5.3.1. However, the following proposition captures an important aspect of this identification.

Proposition 3.40. *Suppose X and Y are locally compact Hausdorff spaces, A is a $C_0(X)$-algebra, $\mathcal{A}$ is its associated upper-semicontinuous bundle, and $\tau : X \to Y$ is continuous. Given $f \in C_c(Y)$ and $a \in A$ define $f \otimes a(y) = f(y)a(\tau(y))$ for all $y \in Y$. Then $f \otimes a \in \Gamma_c(Y, \tau^*\mathcal{A})$ and*

$$C_c(Y) \odot A := \operatorname{span}\{f \otimes a : f \in C_c(Y), a \in A\}.$$

is dense in τ^*A.

Remark 3.41. We will often refer to elements of the form $f \otimes a$ as *elementary tensors*, because, as we will see in Section 5.3.1, they correspond to elementary tensors in a tensor product.

Proof. Given f and a as above view a as a section of the associated bundle. Now define $g(y) = (y, f(y)a(\tau(y)))$. Since everything in sight is continuous, it is clear that $g \in \Gamma(Y, \tau^*\mathcal{A})$. Furthermore, given $y \in Y$ we have $\|g(y)\| = |f(y)|\|a(\tau(y))\|$ so that $\operatorname{supp} g \subset \operatorname{supp} f$. Thus $g \in \Gamma_c(Y, \tau^*\mathcal{A})$. Once we make the identification mentioned in Remark 3.36, this shows $f \otimes a \in \Gamma_c(Y, \tau^*\mathcal{A})$.

We would like to see that the set $C_c(Y) \odot A = \operatorname{span}\{f \otimes a : f \in C_c(Y), a \in A\}$ is dense in $\tau^*A = \Gamma_0(Y, \tau^*\mathcal{A})$. First observe that if $g \in C_0(Y)$, $f \in C_c(Y)$, and $a \in A$ then $g \cdot (f \otimes a) = gf \otimes a$. It follows that $C_c(Y) \odot A$ is closed under the $C_0(Y)$ action. Now suppose $b \in \tau^*\mathcal{A}$. Choose $a \in \mathcal{A}$ so so that $a(\tau(y)) = b$ and $f \in C_c(Y)$ so that $f(y) = 1$. Then $f \otimes a(y) = b$. We can now conclude from Proposition 3.11 that $C_c(Y) \odot A$ is dense in τ^*A. □

Of course, we don't need to be working with pull backs for Proposition 3.40 to hold.

Corollary 3.42. *Suppose A is a $C_0(X)$-algebra, and let $\mathcal{A}$ be is its associated upper-semicontinuous bundle. Given $f \in C_c(X)$ and $a \in A$ define $f \otimes a(x) = f(x)a(x)$ for all $x \in X$. Then $f \otimes a \in \Gamma_c(X, \mathcal{A})$ and*

$$C_c(X) \odot A := \operatorname{span}\{f \otimes a : f \in C_c(X), a \in A\}.$$

is dense in A.

Proof. This result follows immediately from Proposition 3.40 with $\tau = \mathrm{id}$. □

This is a good opportunity to introduce something that will be fundamental to our study of crossed products.

Definition 3.43. Suppose X is a locally compact Hausdorff space and $\mathcal{A}$ is an upper-semicontinuous Banach bundle over X. Given a net $\{f_i\}_{i \in I} \subset \Gamma(X, \mathcal{A})$ and $f \in \Gamma(X, \mathcal{A})$ we say that $f_i \to f$ with respect to the *inductive limit topology* if and only if $f_i \to f$ uniformly and there exists a compact set K in X such that, eventually, all the f_i and f vanish off K. Furthermore, we will say that a function $F : \Gamma(X, \mathcal{A}) \to Y$ is continuous in the inductive limit topology if $F(f_i) \to F(f)$ whenever $f_i \to f$ with respect to the inductive limit topology.

Remark 3.44. First, we will often use Definition 3.43 in the degenerate situation where X is locally compact Hausdorff and $\mathcal{A}$ is the trivial Banach bundle $X \times \mathbb{C}$. In this case there is actually a topology $\mathcal{T}$ on $C_c(X)$ such that a function from $C_c(X)$ into a convex space is continuous with respect to $\mathcal{T}$ if and only if it respects nets which converge in the inductive limit topology [RW98, Lemma D.10]. However, we are not claiming in general that there is actually a topology on $\Gamma(X, \mathcal{A})$ which is characterized by these convergent nets and even in the scalar case there may be nets which converge in $C_c(X)$ with respect to $\mathcal{T}$ and do not satisfy Definition 3.43.

Corollary 3.45. *Suppose X and Y are locally compact Hausdorff spaces, A is a $C_0(X)$-algebra, $\mathcal{A}$ is its associated upper-semicontinuous bundle, and $\tau : X \to Y$ is continuous. Then $C_c(Y) \odot A$ is dense in $\Gamma_c(Y, \tau^*\mathcal{A})$ with respect to the inductive limit topology.*

Proof. Suppose $g \in \Gamma_c(Y, \tau^*\mathcal{A})$. We know from Proposition 3.40 that there exists a net $f_i \in C_c(Y) \odot A$ such that $f_i \to g$ uniformly. Let K be a compact neighborhood of $\operatorname{supp} g$ and choose $\phi \in C_c(Y)$ such that ϕ is one on $\operatorname{supp} g$ and ϕ is zero off K. We showed in the proof of Proposition 3.40 that $C_c(Y) \odot A$ is closed under the $C_0(Y)$ action so that $\phi \cdot f_i \in C_c(Y) \odot A$ for all i. Furthermore, it follows immediately from the fact that $\phi = 1$ on $\operatorname{supp} g$ that we still have $\phi \cdot f_i \to g$ uniformly. Since clearly $\operatorname{supp} \phi \cdot f_i \subset K$ we conclude that $\phi \cdot f_i \to g$ with respect to the inductive limit topology. □

3.2 Groupoid Dynamical Systems

We are finally at a point where we can define what it means to be a groupoid dynamical system. The reason we needed to introduce $C_0(X)$-algebras in the last section is because, just like groupoid actions on spaces, groupoids must act on fibred C^*-algebras.

Definition 3.46. Suppose G is a locally compact Hausdorff groupoid with a Haar system. Let A be a $C_0(G^{(0)})$-algebra and $\mathcal{A}$ its associated upper-semicontinuous bundle. An action α of G on A is a family of functions $\{\alpha_\gamma\}_{\gamma \in G}$ such that,

(a) for each $\gamma \in G$ the map $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$ is an isomorphism,

(b) $\alpha_{\gamma\eta} = \alpha_\gamma \circ \alpha_\eta$ for all $(\gamma, \eta) \in G^{(2)}$, and

(c) $\gamma \cdot a := \alpha_\gamma(a)$ defines a (strongly) continuous action of G on $\mathcal{A}$.

The triple (A, G, α) is called a groupoid dynamical system. We say that (A, G, α) is separable if A is separable and G is second countable.

Remark 3.47. The bundle map associated to an upper-semicontinuous bundle $\mathcal{A}$ is assumed to be open and this is exactly the structure map for the action in Definition 3.46. Thus, as long as G acts on $\mathcal{A}$ continuously, condition (c) above will be satisfied.

Remark 3.48. The assumption that G has a Haar system is not really an integral part of Definition 3.46. However, we don't care about dynamical systems with no Haar system so it's useful to include it as part of the definition.

We will use the following frequently and will not bother to reference it.

Proposition 3.49. *Suppose (A, G, α) is a dynamical system. Then*

(a) $\alpha_u = \mathrm{id}_{A(u)}$ *for all* $u \in G^{(0)}$*, and*

(b) $\alpha_{\gamma^{-1}} = \alpha_\gamma^{-1}$ *for all* $\gamma \in G$.

Proof. Given $u \in G^{(0)}$ we have $\alpha_u = \alpha_{u^2} = \alpha_u \circ \alpha_u$. Since α_u is an automorphism of $A(u)$, the result holds. However, we can now conclude from the fact that $\gamma^{-1}\gamma \in G^{(0)}$ that $\mathrm{id} = \alpha_{\gamma^{-1}\gamma} = \alpha_{\gamma^{-1}} \circ \alpha_\gamma$ and the second half of the proposition follows. □

This next proposition gives an alternate characterization of groupoid dynamical systems which is often easier to work with given that the topology on $\mathcal{A}$ can be poorly behaved. However, there is no way to completely dodge this fact and continuity is almost always the hardest condition to verify.

Proposition 3.50. *Suppose (A, G, α) is a groupoid dynamical system. Then*

$$\alpha(f)(\gamma) = \alpha_\gamma(f(\gamma))$$

defines a $C_0(G)$-linear isomorphism of s^*A onto r^*A.

Conversely, if G is a groupoid, A is a $C_0(G^{(0)})$-algebra, and there is a $C_0(G)$-linear isomorphism $\alpha : s^*A \to r^*A$ then there are isomorphisms $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$ for all $\gamma \in G$. Furthermore, if $\alpha_{\gamma\eta} = \alpha_\gamma \circ \alpha_\eta$ for all $(\gamma, \eta) \in G^{(2)}$ then (A, G, α) is a dynamical system.

Proof. Given $f \in s^*A$ observe that $\alpha_\gamma(f(\gamma)) \in A(r(\gamma))$ for all $\gamma \in G$. Therefore, if we define $\alpha(f)$ as in the statement of the proposition, it follows that $\alpha(f)$ is a section of r^*A. If $\gamma_i \to \gamma$ then $f(\gamma_i) \to f(\gamma)$. It follows from condition (c) of Definition 3.46 that $\alpha_{\gamma_i}(f(\gamma_i)) \to \alpha_\gamma(f(\gamma))$. Thus $\alpha(f) \in \Gamma(G, r^*\mathcal{A})$. Furthermore, each α_γ is an isomorphism so that

$$\|\alpha(f)(\gamma)\| = \|\alpha_\gamma(f(\gamma))\| = \|f(\gamma)\|. \tag{3.3}$$

It immediately follows that $\alpha(f)$ vanishes at infinity if f does. Thus $\alpha(f) \in r^*A = \Gamma_0(G, r^*\mathcal{A})$. Now we will show that α is a $*$-isomorphism. Given $f, g \in s^*A$ and $\gamma \in G$ we have

$$\alpha(f+g)(\gamma) = \alpha_\gamma(f(\gamma) + g(\gamma)) = \alpha(f)(\gamma) + \alpha(g)(\gamma)$$

where the second equality follows from the fact that α_γ is linear. It is just as easy to show that α preserves the rest of the operations. Furthermore, it follows from (3.3) that $\|\alpha(f)\|_\infty = \|f\|_\infty$ and that α is isometric. Lastly, given $f \in r^*A$ define $g \in s^*A$ by $g(\gamma) = \alpha_{\gamma^{-1}}(f(\gamma))$. It is easy enough to see that g is a continuous section

which vanishes at infinity and that $\alpha(g) = f$. Thus α is a $*$-isomorphism. Finally, we compute, for $f \in s^*A$ and $\phi \in C_0(G)$,

$$\alpha(\phi \cdot f)(\gamma) = \alpha_\gamma(\phi \cdot f(\gamma)) = \alpha_\gamma(\phi(\gamma)f(\gamma)) = \phi(\gamma)\alpha_\gamma(f(\gamma)) = \phi \cdot \alpha(f)(\gamma).$$

Now we prove the opposite direction. Suppose that $\alpha : s^*A \to r^*A$ is a $C_0(G)$-isomorphism. It follows from Proposition 3.20 that for each $\gamma \in G$ there is a $*$-isomorphism $\alpha_\gamma : s^*A(\gamma) \to r^*A(\gamma)$ such that $\alpha_\gamma(f(\gamma)) = \alpha(f)(\gamma)$ for all $f \in s^*A$. It then follows from Proposition 3.38 that we can make the identification $s^*A(\gamma) = A(s(\gamma))$ and $r^*A(\gamma) = A(r(\gamma))$. Thus we have satisfied condition (a) of Definition 3.46. Condition (b) is satisfied by assumption. Suppose $\gamma_i \to \gamma$ in G and $a_i \to a$ in $\mathcal{A}$ such that $p(a_i) = s(\gamma_i)$ for all i and $p(a) = s(\gamma)$. Choose $g \in s^*A$ such that $g(\gamma) = a$. Then $\alpha(g) \in r^*A$ and we have, using the fact that both a_i and $g(\gamma_i)$ converge to a,

$$\|\alpha_{\gamma_i}(a_i) - \alpha(g)(\gamma_i)\| = \|\alpha_{\gamma_i}(a_i - g(\gamma_i))\| = \|a_i - g(\gamma_i)\| \to 0.$$

Since $p(\alpha_{\gamma_i}(a_i)) = r(\gamma_i) \to r(\gamma) = p(\alpha_\gamma(a))$ it follows from Proposition 3.10 that $\gamma_i \cdot a_i \to \gamma \cdot a$ and that (A, G, α) is a groupoid dynamical system. □

Remark 3.51. Given a dynamical system we will construct a $*$-algebra structure on $\Gamma_c(G, r^*, \mathcal{A})$. This algebra will eventually be completed into the groupoid crossed product. However, in order to define the convolution operation we will need to use vector valued integration. Most of the time vector valued integrals “just work” and can be treated like scalar valued integrals. Those readers looking for a good reference are referred to [Wil07, Appendix B]. The short version is that given a Radon measure μ on a locally compact Hausdorff space X and a separable Banach algebra B we define $\mathcal{L}^1(X, B)$ to be the set of measurable functions $f : X \to B$ such that[1]

$$\|f\|_1 := \int_X \|f(x)\| d\mu(x) < \infty.$$

Then there is a linear function

$$f \mapsto \int_X f(x) d\mu(x)$$

from $\mathcal{L}^1(X, B)$ into B satisfying

$$\left\| \int_X f(x) d\mu(x) \right\| \leq \int_X \|f(x)\| d\mu(x).$$

Given $b \in B$ and $f \in C_c(X)$ we can define the *elementary tensor* $f \otimes b \in C_c(X, B)$

[1]Since B is separable, this is just the usual definition of measurability.

by $(f \otimes b)(x) = f(x)b$ for all x. We then have

$$\int_X f \otimes b(x) d\mu(x) = \int_X f(x) d\mu(x) b. \tag{3.4}$$

Furthermore, if $L : B \to B'$ is a bounded linear map onto another Banach space B' then

$$L\left(\int_X f(x) d\mu(x)\right) = \int_X L(f(x)) d\mu(x) \tag{3.5}$$

for all $f \in \mathcal{L}^1(X, B)$. In fact (3.5) characterizes the integral when you allow L to be any bounded linear functional. Lastly, there is a Fubini's Theorem for vector valued integrals that is analogous to the scalar one. In particular, when integrating L^1 functions we will freely reorder the integrals.

Remark 3.52. Suppose G is a locally compact Hausdorff groupoid with Haar system λ and A is a $C_0(G^{(0)})$-algebra with associated bundle $\mathcal{A}$. Given $f \in \Gamma_c(G, r^*\mathcal{A})$ it is clear that $f(\gamma) \in A(u)$ for all $\gamma \in G^u$. Therefore, by Remark 3.51 we can form the integral $\int_{G^u} f|_{G^u} d\lambda^u$. Since the support of λ^u is equal to G^u there is no harm denoting this integral by $\int_G f \lambda^u$.

Proposition 3.53. *Suppose (A, G, α) is a dynamical system. Given $f \in \Gamma_c(G, r^*\mathcal{A})$ the function*

$$u \mapsto \int_G f(\gamma) d\lambda^u(\gamma) \tag{3.6}$$

is continuous. Furthermore if $g \otimes a \in C_c(G) \odot A$ then

$$\int_G (g \otimes a)(\gamma) d\lambda^u(\gamma) = \int_G g(\gamma) d\lambda^u(\gamma) a(u)$$

for all $u \in G^{(0)}$.

Proof. It follows from Remark 3.52 that (3.6) is well defined. We will address the second half of the proposition first. Let $g \otimes a$ be an elementary tensor with $g \in C_c(G)$ and $a \in A$. When we restrict $g \otimes a$ to G^u we get a new elementary tensor $g|_{G^u} \otimes a(u) \in C_c(G^u, A(u))$ in the sense of Remark 3.51. However, the result then follows from (3.4).

Now suppose $f \in \Gamma_c(G, r^*\mathcal{A})$, that $u_i \to u$ in $G^{(0)}$, and fix $\epsilon > 0$. Use Corollary 3.45 to find a collection of elementary tensors $\{g_j^i \otimes a_j^i\}$ such that the net $k_j = \sum_i g_j^i \otimes a_j^i$ converges to f with respect to the inductive limit topology. Let K be a compact set which eventually contains the supports of the k_j and f. Next, since K is compact and the λ^u vary continuously, we can find an upper bound M for $\lambda^u(K)$. Now choose J so that $\operatorname{supp} k_J \subset K$ and $\|f - k_J\|_\infty < \epsilon/M$. Then for all $v \in G^{(0)}$ we

have

$$\begin{aligned}\left\|\int_G f d\lambda^v - \int_G k_J d\lambda^v\right\| &= \left\|\int_G f - k_J d\lambda^v\right\| \\ &\leq \int_G \|f - k_J\| d\lambda^v \\ &\leq \|f - k_J\|_\infty \lambda^v(K) < \epsilon.\end{aligned}$$

However, it is clear enough that

$$v \mapsto \int_G g^i_j \otimes a^i_j(\gamma) d\lambda^v(\gamma) = \int_G g^i_j(\gamma) d\lambda^v(\gamma) a^i_j(v)$$

is continuous for all j and i. Since sums of continuous functions are continuous we conclude that

$$v \mapsto \int_G k_J(\gamma) d\lambda^v(\gamma)$$

is continuous as well. It now follows from the previous paragraph, and using the last part of Proposition 3.2, that $\int_G f d\lambda^{u_i} \to \int_G f d\lambda^u$. □

We are now ready to turn $\Gamma_c(G, r^*\mathcal{A})$ into a $*$-algebra. This material is all worked out, in greater generality, in [MW08, Section 4] and many of these proofs are copied from there.

Proposition 3.54. *Let G be a locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}$, A a $C_0(G^{(0)})$-algebra with associated bundle $\mathcal{A}$, and α an action of G on A. Then $\Gamma_c(G, r^*\mathcal{A})$ becomes a $*$-algebra with respect to the operations*

$$f * g(\gamma) = \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma)) d\lambda^{r(\gamma)}(\eta) \quad \textit{and} \quad f^*(\gamma) = \alpha_\gamma(f(\gamma^{-1})^*).$$

Furthermore, these operations are continuous with respect to the inductive limit topology.

We are going to need the following lemma, which is quite similar in nature to Proposition 3.53. It can also be proved in the same fashion, but we have presented a different way of approaching the situation.

Lemma 3.55. *Let $G * G = \{(\gamma, \eta) \in G \times G : r(\gamma) = r(\eta)\}$ and let $r^*\mathcal{A}$ be the pull back of $\mathcal{A}$ to $G * G$. Given $F \in \Gamma_c(G * G, r^*\mathcal{A})$ then*

$$f(\gamma) = \int_G F(\eta, \gamma) d\lambda^{r(\gamma)}(\eta)$$

defines a section in $\Gamma_c(G, r^*\mathcal{A})$.

Proof. First, observe that f is clearly a section. Now, it is straightforward to check that if $F_i \to F$ with respect to the inductive limit topology in $\Gamma_c(G * G, r^*\mathcal{A})$ then $f_i \to f$ with respect to the inductive limit topology. Furthermore, observe that in this case if we show each f_i is continuous and compactly supported then f must be as well. It follows then, that it suffices to show that the lemma holds for elementary tensors $g \otimes a$ where $g \in C_c(G * G)$ and $a \in A$. We can use Lemma 1.71 to extend g to all of $G \times G$. Now, sums of functions of the form $(\gamma, \eta) \mapsto h_1(\gamma)h_2(\eta)$ are dense in $C_c(G \times G)$ with respect to the inductive limit topology [RW98, Corollary B.17], and it is not hard to see that if $f_i \to f$ with respect to the inductive limit topology then $f_i \otimes a \to f \otimes a$ with respect to the inductive limit topology. Thus, using the above argument, we can assume without loss of generality that $g(\gamma, \eta) = h_1(\gamma)h_2(\eta)$ where $h_1, h_2 \in C_c(G)$. However, we now have

$$f(\gamma) = \int_G g(\eta, \gamma)a(r(\gamma))d\lambda^{r(\gamma)}(\eta) = h_2(\gamma)a(r(\gamma)) \int_G h_1(\eta)d\lambda^{r(\gamma)}(\eta).$$

It is clear that in this case f is a continuous compactly supported section, so we are done. □

Proof of Proposition 3.54. First, we have to check that the operations are well defined. This is straightforward for f^* since everything in sight is continuous. The fact that convolution produces a continuous section is exactly Lemma 3.55 once you realize that $(\eta, \gamma) \mapsto f(\eta)\alpha_\eta(\eta^{-1}\gamma)$ is a section in $\Gamma_c(G * G, r^*\mathcal{A})$.

At this point we have to show that the operations are well behaved algebraically. For the most part these computations are omitted. However, we will verify two of them as examples of how the rest should work. First, we will show that the convolution is associative. Suppose $f, g, h \in \Gamma_c(G, r^*\mathcal{A})$ and $\gamma \in G$, then

$$\begin{aligned}
(f * g) * h(\gamma) &= \int_G f * g(\eta)\alpha_\eta(h(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \int_G f(\zeta)\alpha_\zeta(g(\zeta^{-1}\eta))\alpha_\eta(h(\eta^{-1}\gamma))d\lambda^{r(\eta)}(\zeta)d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \int_G f(\zeta)\alpha_\zeta(g(\eta))\alpha_{\zeta\eta}(h(\eta^{-1}\zeta^{-1}\gamma))d\lambda^{s(\zeta)}(\eta)d\lambda^{r(\gamma)}(\zeta)
\end{aligned}$$

where we switched the order of the integrals and used the left invariance of the Haar measure to get the last equality. Continuing the computation by using the fact that α respects the groupoid operations we get

$$(f * g) * h(\gamma) = \int_G \int_G f(\zeta)\alpha_\zeta(g(\eta)\alpha_\eta(h(\eta^{-1}\zeta^{-1}\gamma)))d\lambda^{s(\zeta)}(\eta)d\lambda^{r(\gamma)}(\zeta)$$

$$= \int_G f(\zeta)\alpha_\zeta(g * h(\zeta^{-1}\gamma))d\lambda^{r(\gamma)}(\zeta)$$
$$= f * (g * h)(\gamma).$$

Notice that we used the fact that vector valued integrals are preserved by bounded linear maps to pass the integral through α.

We will also show that $(f * g)^* = g^* * f^*$. Suppose $f, g \in \Gamma_c(G, r^*\mathcal{A})$ and $\gamma \in G$. We have

$$\begin{aligned}
g^* * f^*(\gamma) &= \int_G g^*(\eta)\alpha_\eta(f^*(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \alpha_\eta(g(\eta^{-1})^*)\alpha_\eta(\alpha_{\eta^{-1}\gamma}(f(\gamma^{-1}\eta)^*))d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \alpha_\eta(g(\eta^{-1}))^*\alpha_\gamma(f(\gamma^{-1}\eta))^* d\lambda^{r(\gamma)}(\eta) \\
&= \alpha_\gamma\left(\int_G f(\gamma^{-1}\eta)\alpha_{\gamma^{-1}\eta}(g(\eta^{-1}))d\lambda^{r(\gamma)}(\eta)\right)^*
\end{aligned}$$

To get the last equality we pulled both the α_γ and the * operation out of the integral using the fact that they are bounded linear maps.[2] Now, using left invariance, we get

$$\begin{aligned}
g^* * f^*(\gamma) &= \alpha_\gamma\left(\int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma^{-1}))d\lambda^{s(\gamma)}(\eta)\right)^* \\
&= \alpha_\gamma(f * g(\gamma^{-1})^*) = (f * g)^*(\gamma).
\end{aligned}$$

The rest of the algebraic computations are similar and it is apparent now why we would want to skip them.

The only thing that remains to be verified is that the operations are continuous with respect to the inductive limit topology. This is clearly true for the involution since each α_γ is isometric. The convolution is only slightly more complicated. Suppose $f_i \to f$ and $g_i \to g$ with respect to the inductive limit topology in $\Gamma_c(G, r^*\mathcal{A})$. Let $F_i(\eta, \gamma) = f_i(\eta)\alpha_\eta(g_i(\eta^{-1}\gamma))$ for each i and let $F(\eta, \gamma) = f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))$. Then it is easy to show that $F_i \to F$ with respect to the inductive limit topology in $\Gamma_c(G * G, r^*\mathcal{A})$. For instance, if the support of f_i is eventually contained in the compact set K and the support of g_i is eventually contained in the compact set L then the support of F_i will eventually be contained in $K \times KL$, which is compact. But then $f_i * g_i$ and $f * g$ are defined via integration as in Lemma 3.55. It is now straightforward to show that $f_i * g_i \to f * g$ with respect to the inductive limit

[2] Technically the * operation is conjugate linear but it is linear when viewed as a map into the conjugate algebra.

topology. □

If $f \in \Gamma_c(G, r^*\mathcal{A})$ then $\gamma \mapsto \|f(\gamma)\|$ is upper-semicontinuous and compactly supported. Therefore this function is integrable on G with respect to any Radon measure. This allows us to define the following norm.

Definition 3.56. If (G, A, α) is a dynamical system we define the *I-norm* of $f \in \Gamma_c(G, r^*\mathcal{A})$ to be

$$\|f\|_I = \max\left\{ \sup_{u \in G^{(0)}} \int_G \|f(\gamma)\| d\lambda^u(\gamma),\ \sup_{u \in G^{(0)}} \int_G \|f(\gamma)\| d\lambda_u(\gamma) \right\}.$$

Recall that we define $\lambda_u := (\lambda^u)^{-1}$.

This norm structure interacts nicely with the existing structure on $\Gamma_c(G, r^*\mathcal{A})$. Actually, the I-norm was defined to play along, so to speak. For example, we have to use both supremums in the definition of $\|\cdot\|_I$ in order to make the involution an isometry.

Proposition 3.57. *Suppose (A, G, α) is a dynamical system. Then the I-norm is a norm on $\Gamma_c(G, r^*\mathcal{A})$. Furthermore, for $f, g \in \Gamma_c(G, r^*\mathcal{A})$ we have $\|f * g\|_I \le \|f\|_I \|g\|_I$ and $\|f^*\|_I = \|f\|_I$. Finally, if $f_i \to f$ in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the inductive limit topology then $f_i \to f$ with respect to the I-norm.*

Proof. First we will show that $\|f\|_I < \infty$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$. Since $\operatorname{supp} f$ is compact, and since the λ^u vary continuously, we can find an upper bound M for the set $\{\lambda^u(\operatorname{supp} f)\}$. We can also increase M, if necessary, so that it is also an upper bound for $\{\lambda_u(\operatorname{supp} f)\}$. However, it is now clear that $\|f\|_I \le M\|f\|_\infty$. Showing that $\|\cdot\|_I$ is a norm is straightforward. We will restrict ourselves to showing that it is positive definite. Suppose $f \in \Gamma_c(G, r^*\mathcal{A})$ and $\|f\|_I = 0$. Then, in particular, $\int_G \|f(\gamma)\| d\lambda^u(\gamma) = 0$ for all $u \in G^{(0)}$. Suppose $f(\gamma) \neq 0_{r(\gamma)}$ for some $\gamma \in G$. When restricted to $G^{r(\gamma)}$ the function $\eta \mapsto \|f(\eta)\|$ is positive, continuous, and nonzero at γ. Since $\operatorname{supp} \lambda^{r(\gamma)} = G^{r(\gamma)}$ this implies that $\int_G \|f(\gamma)\| d\lambda^u(\gamma) \neq 0$, which is a contradiction.

Now, suppose $f, g \in \Gamma_c(G, r^*\mathcal{A})$. Then

$$\begin{aligned} \int_G \|f * g(\gamma)\| d\lambda^u(\gamma) &\le \int_G \int_G \|f(\eta)\| \|\alpha_\eta(g(\eta^{-1}\gamma))\| d\lambda^u(\eta) d\lambda^u(\gamma) \\ &= \int_G \|f(\eta)\| \int_G \|g(\eta^{-1}\gamma)\| d\lambda^u(\gamma) d\lambda^u(\eta) \\ &= \int_G \|f(\eta)\| \int_G \|g(\gamma)\| d\lambda^{s(\eta)}(\gamma) d\lambda^u(\eta) \end{aligned}$$

$$\leq \|g\|_I \int_G \|f(\eta)\| d\lambda^u(\eta) \leq \|g\|_I \|f\|_I$$

Similar considerations show that $\int_G \|f * g(\gamma)\| d\lambda_u(\gamma) \leq \|g\|_I \|f\|_I$. It follows that the I-norm is submultiplicative. Next, we compute

$$\begin{aligned}\int_G \|f^*(\gamma)\| d\lambda^u(\gamma) &= \int_G \|\alpha_\gamma(f(\gamma^{-1})^*)\| d\lambda^u(\gamma) \\ &= \int_G \|f(\gamma^{-1})\| d\lambda^u(\gamma) = \int_G \|f(\gamma)\| d\lambda_u(\gamma).\end{aligned}$$

Therefore, it is clear from the definition of the I-norm that $\|f^*\|_I = \|f\|_I$.

Lastly suppose that $f_i \to f$ with respect to the inductive limit topology in $\Gamma_c(G, r^*\mathcal{A})$. Let K be a compact set which eventually contains $\operatorname{supp} f_i$ and $\operatorname{supp} f$. Furthermore, let M be an upper bound for both $\{\lambda^u(K)\}$ and $\{\lambda_u(K)\}$. Fix $\epsilon > 0$ and observe that eventually $\|f - f_i\|_\infty < \epsilon/M$. It follows that, eventually,

$$\begin{aligned}&\int_G \|f(\gamma) - f_i(\gamma)\| d\lambda^u(\gamma) \leq \|f - f_i\|_\infty \lambda^u(K) < \epsilon \quad \text{and,} \\ &\int_G \|f(\gamma) - f_i(\gamma)\| d\lambda_u(\gamma) \leq \|f - f_i\|_\infty \lambda_u(K) < \epsilon.\end{aligned}$$

for all $u \in G^{(0)}$. Thus $\|f_i - f\|_I \to 0$ and we are done. □

Remark 3.58. While the I-norm does make $\Gamma_c(G, r^*\mathcal{A})$ into a $*$-algebra it does not, however, satisfy the C^*-identity. This means that the completion of $\Gamma_c(G, r^*\mathcal{A})$ is not a C^*-algebra. We will instead use the "universal norm" to construct the crossed product in Section 3.4.

Corollary 3.59. *Suppose (A, G, α) is a dynamical system. Then $C_c(G) \odot A$ is dense in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the I-norm.*

Proof. This follows immediately from Corollary 3.45 and the last statement in Proposition 3.57. □

3.3 Covariant Representations

Our goal is to define the notion of a covariant representation of a groupoid dynamical system because these are the representations we will use to define the universal norm. However, in order to do that we first have to discuss the notion of a groupoid representation.

3.3.1 Groupoid Representations

Because groupoids are fibred objects they must be represented on fibred objects. The appropriate bundle in this case is a Borel Hilbert bundle. These objects are relatively classical and only those proofs that seem relevant will be included. All of the necessary technology can be found in [Wil07, Appendix F.1]. In particular, the following definition and remarks are lifted straight from there. Another good reference for this material is [MW08, Section 7].

Remark 3.60. The reader may also wish to consider [Arv76, Chapter 3] where you will learn about analytic and standard spaces. While it is important to understand the the difference between an analytic space and a standard space at some point, beginners are advised to just ignore these Borel considerations on their first pass through the material.

Definition 3.61. Suppose $\mathfrak{H} = \{\mathcal{H}(x)\}_{x\in X}$ is a collection of separable (non-zero) complex Hilbert spaces indexed by an analytic Borel space X. We define the total space to be the disjoint union

$$X * \mathfrak{H} := \{(x,h) : h \in \mathcal{H}(x)\}$$

and let $\pi : X * \mathfrak{H} \to X$ be the obvious projection map. Then $X * \mathfrak{H}$ is an analytic (resp. standard) *Borel Hilbert bundle* if $X * \mathfrak{H}$ has an analytic (resp. standard) Borel structure such that

(a) π is a Borel map and

(b) there is a sequence $\{f_n\}$ of sections such that

  (i) the maps $\bar{f}_n : X * \mathfrak{H} \to \mathbb{C}$ defined by

$$\bar{f}_n(x,h) := (f_n(x), h),$$

  are Borel for each n,

  (ii) for each n and m,

$$x \mapsto (f_n(x), f_m(x))$$

  is Borel, and

  (iii) the functions $\{\bar{f}_n\}$ and π separate points of $X * \mathfrak{H}$.

The sequence $\{f_n\}$ is called a *fundamental sequence* for $X * \mathfrak{H}$. We let $B(X * \mathfrak{H})$ be the set of Borel sections of $X * \mathfrak{H}$.

Remark 3.62. We are using the same notational trickery that is described in Remark 3.36. In particular, a section f of $X * \mathfrak{H}$ is of the form $f(x) = (x, \tilde{f}(x))$ where $\tilde{f}$ maps X into the disjoint union of the $\mathcal{H}(x)$ and $\tilde{f}(x) \in \mathcal{H}(x)$ for all $x \in X$. Of course, f is completely determined by $\tilde{f}$ and just as in Remark 3.36 we will not distinguish between the two functions.

We collect some useful facts from [Wil07, Appendix F] into the next proposition. However, we must first prove a useful lemma, which is an immediate consequence of the Unique Structure Theorem [Arv76, Theorem 3.3.5] and modeled off [Wil07, Lemma D.20].

Lemma 3.63. *Suppose that $(X, \mathcal{B})$ is an analytic Borel space and that $f_n : X \to Y_n$ is a sequence of Borel functions on X which map into countably generated Borel spaces Y_n, and which separate points. Then $\mathcal{B}$ is the smallest σ-algebra in X such that each f_n is Borel. In particular, $g : Y \to X$ is Borel if and only if $f_n \circ g$ is Borel for all n.*

Proof. Let $\mathcal{B}_0$ be the smallest σ-algebra such that each f_n is Borel. Let U_k^n be a countable generating set for the Borel structure on Y_n. Then $\{f_n^{-1}(U_k^n)\}$ is a countable family that generates $\mathcal{B}_0$ and which separates points. Thus $\mathcal{B} = \mathcal{B}_0$ by the Unique Structure Theorem. The rest of the proposition is straightforward. □

The following is little more than a Swiss army knife for dealing with Borel Hilbert bundles, although the fact that all analytic Borel Hilbert bundles are, in some sense, trivial is interesting in its own right.

Proposition 3.64. *Let $X * \mathfrak{H}$ be an analytic Borel Hilbert bundle with fundamental sequence f_n. Then the following are true.*

(a) *We have $f \in B(X * \mathfrak{H})$ if and only if $x \mapsto (f(x), f_n(x))$ is Borel for all n.*

(b) *If $f, g \in B(X * \mathfrak{H})$ then $x \mapsto (f(x), g(x))$ is Borel. It follows that $x \mapsto \|f(x)\|$ is Borel.*

(c) *There exists a fundamental sequence $\{e_k\}$ in $B(X * \mathfrak{H})$ such that*

  (i) *for each $x \in X$ the set $\{e_k(x)\}_k$, minus any possible zero vectors, is an orthonormal basis for $\mathcal{H}(x)$ and*

  (ii) *for each k there is a Borel partition $X = \bigcup_i B_i^k$ and for each (i, k) finitely many Borel functions $\phi_j^{i,k}$ where $1 \le j \le l(i, k)$ such that*

$$e_k(x) = \sum_{j=1}^{l(i,k)} \phi_j^{i,k}(x) f_j(x)$$

for all $x \in B_i^k$.

Such a sequence is called a special orthogonal fundamental sequence.

(d) *There exists a Borel partition $X = X_\infty \cup X_1 \cup X_2 \cup \dots$ of X such that, if $\mathcal{H}_d$ is a fixed Hilbert space of dimension $1 \le d \le \aleph_0$, then $X * \mathfrak{H}$ is Borel isomorphic to the disjoint union $\coprod_{d=1}^{d=\infty} X_d \times \mathcal{H}_d$.*

Remark. These are all results in [Wil07] and we will just reference their locations here. Part (a) is demonstrated in [Wil07, Remark F.3]. In particular it follows from Lemma 3.63 and the fact that the $\bar{f}_n$ and π separate points. (Notice that $\pi \circ f = \mathrm{id}$ is always Borel and that X is countably generated since it's analytic.) Parts (b) and (c) are proved in [Wil07, Proposition F.6] and Part (d) is [Wil07, Corollary F.9]. □

It is worth pointing out that Borel Hilbert bundles do not usually come equipped with an existing Borel structure. Usually we give them one using the following

Proposition 3.65 ([Wil07, Proposition F.8])**.** *Suppose that X is an analytic Borel space and that $\mathfrak{H} = \{\mathcal{H}(x)\}_{x \in X}$ is a family of separable Hilbert spaces. Suppose that $\{f_n\}$ is a countable family of sections of $X * \mathfrak{H}$ such that conditions (ii) and (iii) of axiom (b) in Definition 3.61 are satisfied. Then there is a unique analytic Borel structure on $X * \mathfrak{H}$ such that $X * \mathfrak{H}$ becomes an analytic Borel Hilbert bundle and $\{f_n\}$ is a fundamental sequence.*

Example 3.66. Suppose G is a locally compact Hausdorff groupoid with Haar system λ. Let $L^2(\lambda) := \{L^2(G^u, \lambda^u)\}_{u \in G^{(0)}}$ and form the bundle $G^{(0)} * L^2(\lambda)$. The Borel structure on $G^{(0)} * L^2(\lambda)$ is determined by a sequence of sections $\{\xi_n\}_{n=1}^\infty$ defined as follows. Choose a point separating sequence of functions $\{f_n\}_{n=1}^\infty$ in $C_c(G)$ and define $\xi_n : G^{(0)} \to G^{(0)} * L^2(\lambda)$ by the formula

$$\xi_n(u)(\gamma) = f_n(\gamma) \quad \text{for } \gamma \in G^u.$$

It follows easily from Proposition 3.65 that there is a unique Borel structure on $G^{(0)} * L^2(\lambda)$ such that $G^{(0)} * L^2(\lambda)$ is an analytic Borel Hilbert bundle and ξ_n is a fundamental sequence.

Since Borel Hilbert bundles are fibred objects, they give rise to the following isomorphism groupoid. This groupoid will eventually form the range of a unitary groupoid representation.

Definition 3.67. If $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle then its *isomorphism groupoid* is defined to be

$$\mathrm{Iso}(X * \mathfrak{H}) := \{(x, V, y) : V \in U(\mathcal{H}(y), \mathcal{H}(x))\}$$

equipped with the weakest Borel structure such that $(x, V, y) \mapsto (Vf(y), g(x))$ is Borel for all $f, g \in B(X * \mathfrak{H})$. We define the set of composable pairs to be

$$(\mathrm{Iso}(X * \mathfrak{H}))^{(2)} := \{((x, V, y), (w, U, z)) \in \mathrm{Iso}(X * \mathfrak{H}) \times \mathrm{Iso}(X * \mathfrak{H}) : y = w\}$$

and the operations to be

$$(x, V, y)(y, U, z) := (x, VU, z), \qquad (x, V, y)^{-1} := (y, V^*, z).$$

Of course, we made a number of claims in Definition 3.67 which need to be verified.

Proposition 3.68. *Suppose $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle. Then* $\mathrm{Iso}(X * \mathfrak{H})$ *is a Borel groupoid. Furthermore,* $\mathrm{Iso}(X * \mathfrak{H})$ *is an analytic Borel space and is standard if $X * \mathfrak{H}$ is. Finally, the unit space of* $\mathrm{Iso}(X * \mathfrak{H})$ *can be identified with X and under this identification the range and source maps are given by $r(x, V, y) = x$ and $s(x, V, y) = y$.*

Proof. It is clear from the definition of the groupoid operations that $\mathrm{Iso}(X * \mathfrak{H})$ is a groupoid, that $(x, \mathrm{id}, x) \mapsto x$ is an identification of the unit space with X, and that the range and source maps have the appropriate form. All that remains is to demonstrate the statements concerning the Borel structure.

Recall from Proposition 3.64 that there is a Borel partition $\{X_d\}_{d=0}^{d=\infty}$ of X such that $X * \mathfrak{H}$ is Borel isomorphic to the disjoint union $C = \coprod_{d=0}^{d=\infty} X_d \times \mathcal{H}_d$ where $\mathcal{H}_d$ is a Hilbert space of dimension d. It suffices to see that $\mathrm{Iso}(C)$ has the required properties. However, it is easy to check that $\mathrm{Iso}(C)$ is Borel isomorphic to the disjoint union $\coprod_d \mathrm{Iso}(X_d \times \mathcal{H}_d)$. It is also easy to check that the trivial bundle $\mathrm{Iso}(X_d \times \mathcal{H}_d)$ is Borel isomorphic to the space $X_d \times U(\mathcal{H}_d) \times X_d$ where $U(\mathcal{H}_d)$ is given the (standard) Borel structure coming from the weak operator topology. Thus, $\mathrm{Iso}(X * \mathfrak{H})$ is Borel isomorphic to the disjoint union

$$\coprod_{d=0}^{d=\infty} X_d \times U(\mathcal{H}_d) \times X_d.$$

Now, $U(\mathcal{H}_d)$ has a standard Borel structure. It takes an application of [Arv76, Theorem 3.34], but is otherwise straightforward, to show that a Borel subset of a standard space is standard and a Borel subset of an analytic space is analytic. The upshot is that the X_d are at least analytic and are standard if X is. Thus the product $X_d \times U(\mathcal{H}_d) \times X_d$ is at least analytic and is standard if X is. As a result $\mathrm{Iso}(X * \mathfrak{H})$ is analytic and is standard if X is. Finally, it is easy to see that the groupoid operations are Borel on $X_d \times U(\mathcal{H}_d) \times X_d$ and it follows quickly that they are Borel on the disjoint union. □

The following lemma is useful because it allows us to use a fundamental sequence to check when a map into $\mathrm{Iso}(X * \mathfrak{H})$ is Borel, as opposed to using every section in

$B(X*\mathfrak{H})$. It will be particularly useful for groupoid representations because condition (a) will turn out to be trivial.

Lemma 3.69. *Suppose $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle with fundamental sequence $\{f_n\}$, that Z is some Borel space, and that $U : Z \to \operatorname{Iso}(X * \mathfrak{H})$. Then U is Borel if only if*

- **(a)** *$r \circ U$ is Borel and,*
- **(b)** *$\psi_{n,m} \circ U$ is Borel for all n, m where $\psi_{n,m}(x, V, y) = (Vf_n(y), f_m(x))$.*

Proof. First observe that if $r \circ U$ is Borel then $s \circ U$ is Borel since s is just the composition of r with inversion. Using Lemma 3.63, and noting that X is countably generated since it is analytic, it will suffice to show that $\{\psi_{n,m}\}$, r, and s separate points. Suppose $(x, V, y), (w, U, z) \in \operatorname{Iso}(X * \mathfrak{H})$ such that $r(x, V, y) = r(w, U, z)$, $s(x, V, y) = s(w, U, z)$ and $\psi_{n,m}(x, V, y) = \psi_{n,m}(w, U, z)$ for all n and m. This implies that $x = w$, $y = z$, and that $(V f_n(y), f_m(x)) = (U f_n(y), f_m(x))$ for all n and m. It follows by part (iii) of Definition 3.61 that $V^* f_m(x) = U^* f_m(x)$ for all m. However, given $h \in \mathcal{H}(y)$ we have

$$(Vh, f_m(x)) = (h, V^* f_m(x)) = (h, U^* f_m(x)) = (Uh, f_m(x)).$$

As above, it follows that $Vh = Uh$. This is true for all $h \in \mathcal{H}(y)$ so that $U = V$ and we are done. □

We are almost at the point were we can define a groupoid representation, but first we need to deal with quasi-invariant measures. This will also be our introduction to the modular function which has so far been missing in our treatment of groupoid dynamical systems.

Definition 3.70. Suppose G is a locally compact Hausdorff groupoid with Haar system λ. Given a Radon measure μ on $G^{(0)}$ we define the *induced measures* ν and ν^{-1} to be the Radon measures on G defined by the equations

$$\begin{aligned} \nu(f) &:= \int_{G^{(0)}} \int_G f(\gamma) d\lambda^u(\gamma) d\mu(u), \quad \text{and} \\ \nu^{-1}(f) &:= \int_{G^{(0)}} \int_G f(\gamma) d\lambda_u(\gamma) d\mu(u) \end{aligned}$$

for all $f \in C_c(G)$. We call the measure μ *quasi-invariant* if ν and ν^{-1} are mutually absolutely continuous. In this case we write Δ for the Radon-Nikodym derivative $d\nu/d\nu^{-1}$ and call it the *modular function* of μ. If $\Delta \equiv 1$ ν-almost everywhere then μ is said to be *invariant*.

Remark 3.71. Suppose G is a groupoid with a Haar system λ. Given a measure μ on $G^{(0)}$ we will often denote the measures on G induced from μ by the suggestive notation

$$\nu := \int_G \lambda^u d\mu(u), \qquad \nu^{-1} := \int_G \lambda_u d\mu(u).$$

The reason for the terminology "modular function" is that Δ behaves like the modular function for a locally compact group. We will learn more about this relationship in Section 4.1. Moving on, we can use the following theorem to choose Δ so that it is a groupoid homomorphism from G into $\mathbb{R}_+^\times$.

Theorem 3.72 ([Hah78, Corollary 3.14])**.** *Given a quasi-invariant measure μ on the unit space of a groupoid G with Haar system λ it is possible to choose the modular function of μ, Δ, to be a Borel homomorphism from G to $\mathbb{R}_+^\times$. Moreover, if μ' is another quasi-invariant measure on $G^{(0)}$ that is equivalent to μ so that $\mu' = g\mu$ for a suitable non-negative function g, and if Δ' is the modular function of μ', then $\Delta'(\gamma) = g(r(\gamma))\Delta(\gamma)g(s(\gamma))^{-1}$ ν-almost everywhere, where ν is the measure induced by μ.*

It is not immediately clear that there are such things as quasi-invariant measures. However, the following proposition shows that they are easily constructed. The details are thanks to Dana Williams. It is also proved in [Ren80, Pages 24,25] and detailed in [Muh].

Proposition 3.73. *Suppose G is a second countable locally compact groupoid with Haar system λ. Given a Radon measure μ_0 on $G^{(0)}$ let $\nu_0 = \int_G \lambda^u d\mu_0(u)$ be the measure induced by μ_0. Then ν_0 is a σ-finite measure on G (but not necessarily finite). Let ν be a probability measure on G which is equivalent to ν_0 and define μ to be the image of ν under the source map, i.e. $\mu = s_*\nu$. Then μ is quasi-invariant and, if μ_0 was quasi-invariant to begin with then μ_0 is equivalent to μ.*

Remark 3.74. The measure μ is called the *saturation* of μ_0 and is denoted by $[\mu_0]$.

Proof. To see that ν_0 is σ-finite it suffices to produce $f \in \mathcal{L}^1(G, \nu_0)$ such that $f(\gamma) > 0$ for all $\gamma \in G$. However, since G is second countable, G is σ-compact. Let $G = \bigcup K_n$ with each K_n compact, and let $f_n \in C_c^+(G)$ be such that $f_n(\gamma) > 0$ for all $\gamma \in K_n$. Define

$$\lambda(f_n)(u) = \int_G f_n(\gamma) d\lambda^u(\gamma)$$

for all n. Then $\lambda(f_n) \in C_c^+(G^{(0)})$ and, because μ_0 is a Radon measure,

$$\nu_0(f_n) = \mu(\lambda(f_n)) = \alpha_n < \infty$$

for all n. We may as well assume that we have chosen f_n such that $\|f_n\|_\infty \le 2^{-n}$ and $\alpha_n \le 2^{-n}$. Then $f := \sum_n f_n$ will do.

Now let ν and μ be as in the statement of the proposition and note that, by definition, $\mu(E) = \nu(s^{-1}(E))$. Looking at the characteristic functions, we see that

$$\int_{G^{(0)}} f(u)d\mu(u) = \int_G f(s(\gamma))d\nu(\gamma).$$

For convenience we let ϕ be the Radon-Nikodym derivative of $d\nu/d\nu_0$ and assume, as we can, that $\phi(\gamma) \in (0,\infty)$ for all $\gamma \in G$.

Next, we let $\nu' = \int_G \lambda^u d\mu(u)$. Since $(\nu')^{-1}$ is nothing more than the composition of ν with the inversion map, for the first part of the proof it will suffice to show that if f is a bounded non-negative Borel function on G such that

$$\int_G f(\gamma)d\nu'(\gamma) = 0$$

then

$$\int_G \tilde{f}(\gamma)d\nu'(\gamma) = 0$$

where $\tilde{f}(\gamma) = f(\gamma^{-1})$. We now compute as follows:[3]

$$0 = \int_G f(\gamma)d\nu'(\gamma) = \int_{G^{(0)}} \int_G f(\gamma)d\lambda^u(\gamma)d\mu(u) = \int_G \int_G f(\gamma)d\lambda^{s(\eta)}(\gamma)d\nu(\eta)$$

which, using the definition of ϕ, is

$$\begin{aligned} &= \int_G \int_G f(\gamma)d\lambda^{s(\eta)}(\gamma)\phi(\eta)d\nu_0(\gamma) = \int_{G^{(0)}} \int_G \int_G f(\gamma)\phi(\eta)d\lambda^{s(\eta)}(\gamma)d\lambda^u(\eta)d\mu_0(u) \\ &= \int_{G^{(0)}} \int_G \int_G f(\eta^{-1}\gamma)\phi(\eta)d\lambda^u(\gamma)d\lambda^u(\eta)d\mu_0(u), \end{aligned}$$

where we used left invariance to get the last equality. Now, switch the order of integration, and use left invariance again to get

$$0 = \int_{G^{(0)}} \int_G \int_G f(\eta^{-1})\phi(\gamma\eta)d\lambda^{s(\gamma)}(\eta)d\lambda^u(\gamma)d\mu_0(u) = \int_G \int_G \tilde{f}(\eta)\phi(\gamma\eta)d\lambda^{s(\gamma)}(\eta)d\nu_0(\gamma)$$

[3]Technically we need Proposition 3.109 to do these calculations. However, in the interest of not getting sidetracked we will postpone these considerations until the next section.

which, using the fact that $\phi^{-1} = d\nu_0/d\nu$, is

$$0 = \int_G \int_G \tilde{f}(\eta)\phi(\gamma\eta)\phi(\gamma)^{-1} d\lambda^{s(\gamma)}(\eta) d\nu(\gamma). \tag{3.7}$$

Notice that, since f is non-negative and $\phi(\gamma) > 0$ for all γ, we have

$$\int_G \tilde{f}(\eta)\phi(\eta\gamma)\phi(\gamma)^{-1} d\lambda^{s(\gamma)}(\eta) = 0 \tag{3.8}$$

if and only if

$$\int_G \tilde{f}(\eta) d\lambda^{s(\gamma)}(\eta) = 0. \tag{3.9}$$

At this point it follows from (3.7) that there exists a ν-null set N such that $\gamma \neq N$ implies that (3.8) holds. Clearly N is s-saturated: $s^{-1}(s(N)) = N$. Thus $s(N)$ is μ-null and, since (3.8) holds if and only if (3.9) does, we've established that

$$\int_G \tilde{f}(\eta) d\lambda^u(\eta) = 0$$

for μ-almost all u. But then

$$\int_G \tilde{f}(\eta) d\nu'(\eta) = \int_{G^{(0)}} \int_G \tilde{f}(\eta) d\lambda^u(\eta) d\mu(u) = 0.$$

For the second assertion suppose that μ_0 is quasi-invariant. But then if f is a bounded non-negative Borel function on $G^{(0)}$

$$\mu(f) = \int_G f(s(\gamma)) d\nu(\gamma) = \int_G f(s(\gamma))\phi(\gamma) d\nu_0(\gamma)$$

which, using the quasi-invariance of μ_0, is

$$\begin{aligned} &= \int_G f(r(\gamma))\phi(\gamma^{-1})\Delta(\gamma) d\nu_0(\gamma) = \int_{G^{(0)}} \int_G f(r(\gamma))\phi(\gamma^{-1})\Delta(\gamma) d\lambda^u(\gamma) d\mu_0(u) \\ &= \int_{G^{(0)}} f(u) \left(\int_G \phi(\gamma^{-1})\Delta(\gamma) d\lambda^u(\gamma) \right) d\mu_0(u). \end{aligned}$$

Since

$$\alpha(u) := \int_G \phi(\gamma^{-1})\Delta(\gamma) d\lambda^u(\gamma)$$

is a non-negative, (extended) real-valued function it follows that $\mu \ll \mu_0$. The argument is symmetric in μ and μ_0 so $\mu_0 \ll \mu$ and μ is equivalent to μ_0. □

Being able to restrict ourselves to finite quasi-invariant measures will be useful later on.

Corollary 3.75. *Suppose G is a second countable locally compact Hausdorff groupoid with a Haar system. Then every quasi-invariant measure on $G^{(0)}$ is equivalent to a finite quasi-invariant measure.*

Proof. This follows immediately from Proposition 3.73 since the measure μ constructed there is finite. □

We are finally ready to define the notion of a unitary representation of a groupoid. It is perhaps notable that, unlike the group case, groupoid representations are Borel creatures.

Definition 3.76. Let G be a locally compact Hausdorff groupoid with a Haar system. A *groupoid representation* of G is a triple $(\mu, G^{(0)} * \mathfrak{H}, L)$ where μ is a finite quasi-invariant measure on $G^{(0)}$, $G^{(0)} * \mathfrak{H}$ is an analytic Borel Hilbert bundle, and $L : G \to \operatorname{Iso}(G^{(0)} * \mathfrak{H})$ is a Borel homomorphism such that $L(\gamma) = (r(\gamma), L_\gamma, s(\gamma))$ for some unitary $L_\gamma : \mathcal{H}(s(\gamma)) \to \mathcal{H}(r(\gamma))$.

Remark 3.77. We have taken the quasi-invariant measure μ to be finite. The reason we have done this is because it is convenient. However, there is no harm in working with σ-finite quasi-invariant measures either. Any such measure is, at least in the second countable case, equivalent to a finite one by Corollary 3.75 and Remarks 3.83 and 3.87 address how the the theory could be expanded to include the σ-finite case using this fact.

Example 3.78. Let μ be a finite quasi-invariant measure on the unit space of a second countable locally compact Hausdorff groupoid G equipped with a Haar system λ. Let $G^{(0)} * L^2(\lambda)$ be as in Example 3.66. Define $L : G \to \operatorname{Iso}(G^{(0)} * L^2(\lambda))$ by $L(\gamma) = (r(\gamma), L_\gamma, s(\gamma))$ where $L_\gamma : L^2(G^{s(\gamma)}, \lambda^{s(\gamma)}) \to L^2(G^{r(\gamma)}, \lambda^{r(\gamma)})$ is defined by the formula

$$(L_\gamma \xi)(\eta) = \xi(\gamma^{-1}\eta).$$

All we really need to show is that L is a Borel map. We will use Lemma 3.69. First, observe that $r \circ L = r$ is clearly Borel. We must show that the function

$$\psi_{n,m}(\gamma) = (L_\gamma \xi_n(s(\gamma)), \xi_m(r(\gamma))) = \int_G \overline{\xi_n(s(\gamma))(\gamma^{-1}\eta)} \xi_m(r(\gamma))(\eta) d\lambda^{r(\gamma)}(\eta)$$

is Borel for all n, m where ξ_n is the fundamental sequence constructed in Example 3.66. However, the ξ_n are given by $\xi_n(u)(\gamma) = f_n(\gamma)$ where f_n is a continuous compactly supported function for each n. So in this case $\psi_{n,m}$ is actually continuous for all n, m. Thus the triple $(\mu, G^{(0)} * L^2(\lambda), L)$ forms a representation of G. This type of representation is called the *left regular representation* associated to μ.

3.3.2 Decomposable Representations

Now that we have covered the groupoid half of a covariant representation we have to deal with the C^*-algebraic half. Unfortunately this is not entirely straightforward since we have to deal with the fibred structure on the algebras and this means working with decomposable operators and representations. We will not present a self contained exposition of decomposable operators here. Instead we will only provide those proofs which seem relevant. For a more complete treatment the reader is referred to [Wil07, Section F.3] and [Arv76, Chapter 4].

Remark 3.79. We will always assume, unless explicitly stated otherwise, that all of our representations are nondegenerate.

We begin by forming a Hilbert space out of a given Borel Hilbert bundle. The definition is fairly straightforward.

Definition 3.80. Suppose $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and μ is a measure on X. Let

$$\mathcal{L}^2(X * \mathfrak{H}, \mu) = \{f \in B(X * \mathfrak{H}) : x \mapsto \|f(x)\|^2 \text{ is integrable}\}$$

and give $\mathcal{L}^2(X * \mathfrak{H}, \mu)$ the operations of pointwise addition and scalar multiplication. Let $L^2(X * \mathfrak{H}, \mu)$ be the quotient of $\mathcal{L}^2(X * \mathfrak{H}, \mu)$ where functions which agree μ-almost everywhere are identified. Equipped with the operations coming from $\mathcal{L}^2(X * \mathfrak{H}, \mu)$ and the inner product

$$(f, g) = \int_X (f(x), g(x))d\mu(x)$$

$L^2(X * \mathfrak{H}, \mu)$ becomes a Hilbert space known as the *direct integral* of $\mathfrak{H}$ with respect to μ.

Remark 3.81. The fact that the integral even makes sense follows from Proposition 3.64 and an application of the Cauchy-Schwartz inequality. The rest of the assertions made in Definition 3.80 are straightforward to verify. In any case, they are all addressed in [Wil07, Appendix F.2]. It is worthwhile to point out that the direct integral is classically denoted

$$\int_X^{\oplus} \mathcal{H}(x)d\mu(x).$$

The following proposition, which we cite without proof, guarantees that we will not have to deal with any nonseparable weirdness.

Proposition 3.82 ([Wil07, Lemma F.17])**.** *If $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and μ is a finite Borel measure on X then $L^2(X * \mathfrak{H}, \mu)$ is a separable Hilbert space.*

Remark 3.83. Proposition 3.82 is one reason why we chose to restrict ourselves to finite quasi-invariant measures in Definition 3.76. However, as noted in Remark 3.77, if μ is σ-finite then it is equivalent to a finite measure. It is easy to see that, up to unitary equivalence, $L^2(X * \mathfrak{H}, \mu)$ only depends on the equivalence class of μ. Thus, $L^2(X * \mathfrak{H}, \mu)$ is separable in the σ-finite case as well.

We take a moment to discuss pull back Borel Hilbert bundles.

Example 3.84 ([[Wil07, Example F.18]). Suppose that $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle with fundamental sequence $\{f_n\}$ and that $\sigma : Y \to X$ is a Borel map. Then we can form the pull back Borel Hilbert bundle

$$\sigma^*(X * \mathfrak{H}) := \{(y, h) : h \in \mathcal{H}(\sigma(y))\}.$$

We use Proposition 3.65 to give $\sigma^*(X * \mathfrak{H})$ the Borel structure coming from the fundamental sequence $\{f_n \circ \sigma\}$. It follows that $f \in B(X * \mathfrak{H})$ implies that $f \circ \sigma \in B(\sigma^*(X * \mathfrak{H}))$. If ν is a finite Borel measure on Y and if $\sigma_*\nu$ is the push forward measure on X then it turns out that $W(f)(y) = f(\sigma(y))$ defines an isometry

$$W : L^2(X * \mathfrak{H}, \sigma_*\nu) \to L^2(\sigma^*(X * \mathfrak{H}), \nu)$$

which is an isomorphism if σ is a Borel isomorphism.

Moving on, the basic idea will be that certain representations of C^*-algebras on the direct integral $L^2(X * \mathfrak{H}, \mu)$ can be decomposed into representations on the fibres $\mathcal{H}(x)$. In order to make sense out of this we will eventually need the following

Definition 3.85. Suppose that $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and μ is a finite measure on X. An operator T on $L^2(X * \mathfrak{H}, \mu)$ is called *diagonal* if there is a bounded Borel function $\phi \in \mathcal{B}^b(X)$ such that

$$Th(x) = \phi(x)h(x)$$

for μ-almost every x. The collection of diagonal operators on $L^2(X * \mathfrak{H}, \mu)$ is denoted by $\Delta(X * \mathfrak{H}, \mu)$. If $\phi \in \mathcal{B}^b(X)$ then the associated diagonal operator is denoted by T_ϕ.

Proposition 3.86. *[Wil07, Lemma F.15] Suppose that $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and that μ is a finite Borel measure on X. Then $\Delta(X * \mathfrak{H}, \mu)$ is an abelian von Neumann subalgebra of $B(L^2(X * \mathfrak{H}, \mu))$, and the map $\phi \mapsto T_\phi$ induces an isomorphism of $L^\infty(X, \mu)$ onto $\Delta(X * \mathfrak{H}, \mu)$.*

Remark 3.87. As discussed, we are assuming that the measure μ is finite, but these definitions and their theory can be extended to σ-finite measures by again using the fact that every σ-finite measure is equivalent to a finite one. We actually do this for

the theory of direct integrals of operators and representations.[4] The key fact is that the unitary induced by the Radon-Nikodym derivative of two equivalent measures will intertwine the diagonal operators, and as such will respect the theory of decomposable operators. In short, the finiteness of μ will not be an essential part of the theory of covariant representations and is, as stated in Remark 3.77, a convenience.

Now, before we can decompose representations on $L^2(X * \mathfrak{H})$ we have to be able to decompose operators. In order to do this we must have some idea of what happens after such a decomposition.

Definition 3.88. Suppose $X*\mathfrak{H}$ is an analytic Borel Hilbert bundle with fundamental sequence $\{f_n\}$. A family of bounded linear maps $T(x) : \mathcal{H}(x) \to \mathcal{H}(x)$ is a *Borel field of operators* if

$$x \mapsto (T(x)f_n(x), f_m(x))$$

is Borel for all n and m.

Of course, we would like to see how this relates to the Borel structure on $X * \mathfrak{H}$.

Proposition 3.89. *Suppose $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and that we have a family of bounded linear maps $T(x) : \mathcal{H}(x) \to \mathcal{H}(x)$. We can define a bundle map $\widehat{T} : X * \mathfrak{H} \to X * \mathfrak{H}$ such that $\widehat{T}(x, h) = (x, T(x)h)$. Then $\widehat{T}$ is Borel if and only if $\{T(x)\}$ is a Borel field of operators.*

Remark 3.90. Thus, a Borel field of operators is essentially nothing more than an endomorphism of a Borel Hilbert bundle.

Proof. Let e_l be a special orthogonal fundamental sequence for $X * \mathfrak{H}$. Suppose $\widehat{T}$ is Borel. Then $\bar{e}_k \circ \widehat{T} \circ e_l$ is clearly Borel and, tracing through the definitions,

$$\overline{\bar{e}_k \circ \widehat{T} \circ e_l(x)} = \overline{\bar{e}_k(x, T(x)e_l(x))} = (T(x)e_l(x), e_k(x)). \tag{3.10}$$

For the reverse direction, suppose $x \mapsto (T(x)e_l(x), e_k(x))$ is Borel for all l and k. We want to show that $\bar{e}_k \circ \widehat{T}$ is Borel for all k. Well, using the Fourier Identity,

$$\bar{e}_k \circ \widehat{T}(x, h) = (e_k(x), T(x)h) = \sum_l (e_l(x), h)\overline{(T(x)e_l(x), e_k(x))}.$$

However, $(x, h) \mapsto (e_l(x), h)$ is Borel since e_l is a fundamental sequence and $x \mapsto (T(x)e_l(x), e_k(x))$ is Borel by assumption. This suffices to show that $\bar{e}_k \circ \widehat{T}$ is Borel and we are done. $\square$

[4]For the curious reader, this was actually added in after much of the thesis was finished because it is needed in Section 6.3.

Given a Borel field of bounded linear operators we would like to be able to glue them together to form a bounded linear operator on the direct integral of the Hilbert spaces.

Proposition 3.91. *Suppose $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and that μ is a σ-finite measure on X. Let $\{T(x)\}$ be a Borel field of bounded linear operators such that*

$$\Lambda := \operatorname*{ess\,sup}_{x \in X} \|T(x)\| < \infty.$$

Then there exists a bounded linear operator $T \in B(L^2(X * \mathfrak{H}, \mu))$ defined by $Tf(x) = T(x)f(x)$ for all $f \in \mathcal{L}^2(X * \mathfrak{H}, \mu)$ such that $\|T\| = \Lambda$.

Remark 3.92. The classical notation for the operator defined in Proposition 3.91 is

$$\int_X^{\oplus} T(x) d\mu(x).$$

This, of course, is in line with the direct integral notation for Borel Hilbert bundles. We will also occasionally refer to T as the *direct integral* of the $T(x)$.

Proof. Let us start by supposing that μ is actually a finite measure. The first thing we have to do is make sure that everything is well defined. It is not too difficult to show that if S is the countable family of rational linear combinations of some special orthogonal fundamental sequence for $X * \mathfrak{H}$ then $\{h(x) : h \in S\}$ is dense in $\mathcal{H}(x)$ for all $x \in X$. This implies that

$$\|T(x)\| = \sup_{h \in S, h(x) \neq 0} \|T(x)h(x)\| \|h(x)\|^{-1}.$$

It then follows that the function $x \mapsto \|T(x)\|$ is Borel and that taking the essential supremum Λ makes sense.

Given a Borel field of operators and $f \in B(X * \mathfrak{H})$ we have $\widehat{T} \circ f(x) = T(x)f(x)$ for all $x \in X$. Since $\widehat{T}$ is Borel, we have $\widehat{T} \circ f \in B(X * \mathfrak{H})$. Now, define T on $\mathcal{L}^2(X * \mathfrak{H}, \mu)$ as in the statement of the proposition and observe that $Tf = \widehat{T} \circ f$ so that Tf is a Borel section. Furthermore $\|Tf(x)\| \leq \Lambda \|f(x)\|$ μ-almost everywhere so, because $x \mapsto \|f(x)\|^2$ was integrable, $x \mapsto \|Tf(x)\|^2$ must be integrable as well. Hence $Tf \in \mathcal{L}^2(X * \mathfrak{H}, \mu)$. It is straightforward to show that T is a linear map. Finally

$$\|Tf\|^2 = \int_X \|T(x)f(x)\|^2 d\mu(x) \leq \Lambda^2 \int_X \|f(x)\|^2 d\mu(x) = \Lambda^2 \|f\|^2$$

so that T factors to an operator on $L^2(X * \mathfrak{H})$ and is bounded with norm at most Λ.

Next we will show the reverse inequality. First, because of the last part of Proposition 3.64 we can assume without loss of generality that $X * \mathfrak{H}$ is trivial. If not, decompose $X * \mathfrak{H}$ as a disjoint union of trivial bundles $X * \mathfrak{H} = \coprod_{d=0}^{\infty} X_d \times \mathcal{H}_d$. Then if we show that $\operatorname{ess\,sup}_{x \in X_d} \|T(x)\| \leq \|T\|$ for all d it will follow that $\Lambda \leq \|T\|$. So assume that $X * \mathfrak{H} = X \times \mathcal{H}$ is trivial and note that $L^2(X * \mathfrak{H}, \mu) = L^2(X, \mathcal{H}, \mu)$. Suppose $h \in \mathcal{H}$ is a unit vector and observe that, since μ is finite, we can view h as a constant function in $L^2(X, \mathcal{H}, \mu)$. It follows that $x \mapsto \|T(x)h\|$ is a scalar function in $L^2(X, \mu)$. Now, take $f \in \mathcal{L}^2(X, \mu)$ such that $\|f\|_2 = 1$. Observe that we can define the function $f \otimes h(x) = f(x)h$ for all $x \in X$, that $f \otimes h \in L^2(X, \mathcal{H})$, and $\|f \otimes h\|_2 = 1$. It follows that

$$\int_X \|T(x)h\|^2 |f(x)|^2 d\mu(x) = \int_X \|T(f \otimes h)(x)\|^2 d\mu(x) = \|T(f \otimes h)\|^2 \leq \|T\|^2. \quad (3.11)$$

We then conclude from the following general nonsense that $\|T(x)h\| \leq \|T\|$ everywhere off a μ-null set N_h. Let $\phi(x) = \|T(x)h\|^2$ and define M_ϕ to be the multiplication operator on $L^1(X)$. Now, given $k \in L^1(X)$ such that $\|k\|_1 = 1$ let $f(x) = \sqrt{|k(x)|}$ and observe that $f \in L^2(X)$ with $\|f\|_2 = 1$. It follows from (3.11) that $\|M_\phi k\|_1 \leq \|T\|^2$. Hence $\|\phi\|_\infty = \|M_\phi\| \leq \|T\|^2$. Thus, as required, there exists a μ-null set N_h such that $x \notin N_h$ implies $\|T(x)h\| \leq \|T\|$. Next, let S be the set of all rational linear combinations of a countable basis for $\mathcal{H}$ such that $\|h\| = 1$ for all $h \in S$. We can find a new μ-null set $N = \bigcup_{h \in S} N_h$ such that given $x \notin N$ we have $\|T(x)h\| \leq \|T\|$ for all $h \in S$. It follows that $\|T(x)\| \leq \|T\|$ μ-almost everywhere and we are done.

Now suppose μ is σ-finite. Then in the usual fashion we can find a finite measure ν which is equivalent to μ. Let $d\nu/d\mu$ be the Radon-Nikodym derivative and define $U : L^2(X * \mathfrak{H}, \mu) \to L^2(X * \mathfrak{H}, \nu)$ by $Uf(x) = (d\mu/d\nu)^{1/2}(x)f(x)$. It is straightforward to show that U is a unitary. Since ν is finite we can form the direct integral

$$T' = \int_X^{\oplus} T(x) d\nu(x)$$

on $L^2(X * \mathfrak{H}, \nu)$. This allows us to define an operator T on $L^2(X * \mathfrak{H}, \mu)$ by $T = U^* T' U$. We immediately have $\|T\| = \|T'\| = \Lambda$. Furthermore, we can compute

$$\begin{aligned} Tf(x) = U^* T' U f(x) &= \left(\frac{d\nu}{d\mu}(x)\right)^{1/2} T' U f(x) = \left(\frac{d\nu}{d\mu}(x)\right)^{1/2} T(x) U f(x) \\ &= \left(\frac{d\nu}{d\mu}(x)\right)^{1/2} T(x) \left(\frac{d\mu}{d\nu}(x)\right)^{1/2} f(x) = T(x) f(x). \end{aligned} \qquad \square$$

We can now "integrate" operators and will shortly describe how to integrate representations. Our eventual goal will be to show that $C_0(X)$-linear representations all

have this form.

Proposition 3.93. *Suppose X is a second countable locally compact Hausdorff space, A is a separable $C_0(X)$-algebra, $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and μ is a σ-finite measure on X. Given a collection of representations $\pi_x : A(x) \to B(\mathcal{H}(x))$ for all $x \in X$ such that given $a \in A$ the set $\{\pi_x(a(x))\}$ is a Borel field of operators then we can form a representation*

$$\pi = \int_X^{\oplus} \pi_x d\mu(x)$$

of A on $L^2(X * \mathfrak{H}, \mu)$ called the direct integral *and defined for $a \in A$ by*

$$\pi(a) = \int_X^{\oplus} \pi_x(a(x))d\mu(x).$$

Remark 3.94. Of course, $\pi = \int_X^{\oplus} \pi_x d\mu(x)$ is just the classic direct integral notation of representations. In fact, as long as you are willing to confuse π_x with its lift to A, Proposition 3.93 and the upcoming Proposition 3.99 are just the classic theory of decomposable representations.

Proof. Let $\{\pi_x\}$ be as above. Given $a \in A$ we assumed that $\{\pi_x(a(x))\}$ is a Borel field of operators. Since $\|\pi_x(a(x))\| \leq \|a(x)\| \leq \|a\|$ for all $x \in X$, it follows that $\{\pi_x(a(x))\}$ is bounded by $\|a\|$. Therefore we can form the direct integral

$$\pi(a) = \int_X^{\oplus} \pi_x(a(x))d\mu(x).$$

All we need to do is show that π preserves the algebraic operations. However, given $f \in L^2(X * \mathfrak{H}, \mu)$ we have

$$\pi(a+b)f(x) = \pi_x((a+b)(x))f(x) = \pi_x(a(x))f(x)+\pi_x(b(x))f(x) = (\pi(a)+\pi(b))f(x).$$

It is just as easy to show that the rest of the operations are preserved. $\square$

Now that we know what "integrated" operators and representations look like we can define the decomposable ones.

Definition 3.95. Given an analytic Borel Hilbert bundle $X * \mathfrak{H}$ and a finite measure μ on X, an operator $T \in B(L^2(X * \mathfrak{H}, \mu))$ is called *decomposable* if there exists an essentially bounded Borel field of operators $\{T(x)\}_{x \in X}$ such that

$$T = \int_X^{\oplus} T(x)d\mu(x).$$

This lemma shows that the decomposition of an operator is unique almost everywhere.

Lemma 3.96 ([Wil07, Lemma F.20])**.** *Suppose that $X * \mathfrak{H}$ is a Borel Hilbert bundle and that μ is a finite measure on X. Let $\{T(x)\}$ be an essentially bounded Borel field of operators and let T be the direct integral operator on $L^2(X * \mathfrak{H}, \mu)$.*

(a) *If $T = 0$ then $T(x) = 0$ for μ-almost all x.*

(b) *If $\{T'(x)\}$ is another essentially bounded Borel field of operators such that $T = \int_X^{\oplus} T'(x)d\mu(x)$ then $T'(x) = T(x)$ for μ-almost all x.*

Proof. Clearly it suffices to prove part (a). Let $\{e_l\}$ be a special orthogonal fundamental sequence. Then $e_l \in L^2(X * \mathfrak{H}, \mu)$ for all l and for all l and k we have

$$(T(x)e_l(x)|e_k(x)) = 0 \quad \text{for } \mu\text{-almost all } x.$$

It follows that $T(x) = 0$ almost everywhere. $\square$

Now, if T is the direct integral of a Borel field of operators $\{T(x)\}$ then, for all $\phi \in L^\infty(X, \mu)$,

$$TT_\phi(f)(x) = \phi(x)T(x)f(x) = T_\phi Tf(x).$$

Thus, T is in the commutant of the von Neumann algebra $\Delta(X * \mathfrak{H}, \mu)$. It is a deep result that this characterizes the decomposable operators.

Theorem 3.97 ([Dix81, II.2.5 Theorem 1])**.** *Suppose that $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and that μ is a finite measure on X. Let $T \in B(L^2(X * \mathfrak{H}, \mu))$. Then T is decomposable if and only if T is in the commutant of the diagonal operators $\Delta(X * \mathfrak{H}, \mu)$.*

The reason we went through all of this is that we would like to use this decomposition theorem to decompose certain representations of $C_0(X)$-algebras so that they are given by direct integrals as in Proposition 3.93.

Definition 3.98. Suppose X is a second countable locally compact Hausdorff space, A is a $C_0(X)$-algebra, $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and μ is a finite Borel measure on X. We say a representation $\pi : A \to B(L^2(X * \mathfrak{H}, \mu))$ is $C_0(X)$-*linear* if

$$\pi(\phi \cdot a) = T_\phi \pi(a).$$

for all $a \in A$ and $\phi \in C_0(X)$.

This definition tells us what kind of representations are decomposable, so now lets decompose them.

Proposition 3.99. *Suppose X is a second countable locally compact Hausdorff space, A is a separable $C_0(X)$-algebra, $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle and μ is a finite Borel measure on X. Given a $C_0(X)$-linear representation π of A on $L^2(X * \mathfrak{H}, \mu)$ there exists possibly degenerate representations $\pi_x : A(x) \to B(\mathcal{H}(x))$ such that given $a \in A$ the set $\{\pi_x(a(x))\}$ is an essentially bounded Borel field of operators and*

$$\pi = \int_X^{\oplus} \pi_x d\mu(x)$$

Furthermore, the representations π_x are nondegenerate μ-almost everywhere and are uniquely determined up to a μ-null set.

Remark 3.100. Given π and A as above we will generally refer to $\{\pi_x\}$ as a decomposition of π and call π the direct integral of $\{\pi_x\}$.

Proof. Suppose $a \in A$ and let Φ_A implement the $C_0(X)$-action on A. Then

$$T_\phi \pi(a) = \pi(\Phi_A(\phi)a) = \pi(\Phi_A(\overline{\phi})a^*)^* = (T_{\overline{\phi}}\pi(a^*))^* = \pi(a)T_\phi$$

where we used the fact that $\Phi_A(\phi)$ is in the center of the multiplier algebra. Thus $\pi(a) \in (\Delta(X * \mathfrak{H}))'$ and it follows from Theorem 3.97 that there exists a Borel field of bounded linear operators $\{\pi'_x(a)\}$ such that $\pi(a) = \int_X^{\oplus} \pi'_x(a)d\mu(x)$.

Define $\pi'_x : A \to B(\mathcal{H}(x))$ such that $\pi'_x(a)$ is given by the above decomposition. We would be done if not for the fact that each π'_x is only a representation "almost everywhere." Now, it uses the separability of A in a fundamental way, but we can actually make our choices so that each π'_x is a $*$-homomorphism. The details are worked out for a slightly different context in [Arv76, Section 4.2] and have also been included here. Suppose $a, b \in A$. It is clear that $\{\pi'_x(a) + \pi'_x(b)\}$ is an essentially bounded Borel field of operators. As such we have the direct integral

$$T = \int_X^{\oplus} \pi'_x(a) + \pi'_x(b) d\mu(x).$$

Furthermore, it is clear that for all $f \in L^2(X * \mathfrak{H})$ we have

$$Tf(x) = \pi'_x(a)f(x) + \pi'_x(b)f(x) = \pi(a)f(x) + \pi(b)f(x).$$

Thus $T = \pi(a) + \pi(b) = \pi(a+b)$ and $\{\pi'_x(a) + \pi'_x(b)\}$ is another decomposition for $\pi(a+b)$. It follows from Lemma 3.96 that there exists a μ-null set $N_{a,b}$ such that for all $x \notin N_{a,b}$ we have $\pi'_x(a+b) = \pi'_x(a) + \pi'_x(b)$. Now, by enlarging $N_{a,b}$ we can use similar arguments to assume that for $x \notin N_{a,b}$ we have

$$\pi'_x(a+b) = \pi'_x(a) + \pi'_x(b), \tag{3.12}$$

$$\pi'_x(ab) = \pi'_x(a)\pi'_x(b), \tag{3.13}$$
$$\pi'_x(a^*) = \pi'_x(a)^*, \quad \text{and} \tag{3.14}$$
$$\pi'_x(\lambda a) = \lambda\pi'_x(a) \quad \text{for all } \lambda \in \mathbb{Q} + i\mathbb{Q}. \tag{3.15}$$

Let $\{a_i\}$ be a countable dense sequence in A and let $N = \bigcup_{i,j} N_{a_i,a_j}$. Then N is a μ-null set and for all $x \notin N$ we know that the identities (3.12)-(3.15) hold for any elements of $\{a_i\}$. Let S be the countable family of all finite sums of elements of the form $rb_1 \cdots b_n$ where $r \in \mathbb{Q} + i\mathbb{Q}$ and $b_1, \ldots, b_n \in \{a_i\} \cup \{a_i^*\}$. It is straightforward to show that, for all $x \notin N$, (3.12)-(3.15) extend to all of S. It follows from Proposition 3.91 that given $a \in S$ we have $\|\pi'_x(a)\| \leq \|\pi(a)\| \leq \|a\|$ for μ-almost all x. Thus, by enlarging N by another countable family of null sets, we can assume that given $x \notin N$ we have

$$\|\pi'_x(a)\| \leq \|a\|$$

for all $a \in S$. Thus π'_x is a bounded $*$-homomorphism on the normed $*$-algebra S. Therefore, given $x \notin N$ we can extend π'_x to a $*$-homomorphism $\tilde{\pi}_x : A \to B(\mathcal{H}(x))$. If $x \in N$ we let $\tilde{\pi}_x$ be the trivial representation on $\mathcal{H}(x)$.

Now we must show that $\{\tilde{\pi}_x(a)\}$ is also a decomposition of $\pi(a)$ for all $a \in A$. Suppose $a \in A$, that $a_{i_j} \to a$, and that $\{f_n\}$ is a fundamental sequence. Fix n and m and let $\phi : X \to \mathbb{C}$ be defined by $\phi(x) = (\tilde{\pi}_x(a)f_n(x)|f_m(x))$ and $\phi_j : X \to \mathbb{C}$ be defined by $\phi_j(x) = (\pi'_x(a_{i_j})f_n(x)|f_m(x))$. Then for $x \notin N$ we have $\phi(x) = \lim_j \phi_j(x)$. Since the pointwise limit of Borel functions is Borel, it follows that $\phi|_N$ is Borel. However, $\phi(N) = 0$ so that ϕ must be Borel everywhere. It follows that $\{\tilde{\pi}_x(a)\}$ is a Borel field of operators that is clearly bounded by $\|a\|$. Now, given $a \in A$ let $\{a_{i_j}\}$ be a sequence in $\{a_i\}$ converging to a. It is not difficult to use the fact that $\tilde{\pi}_x$ is an extension of π'_x from S and the fact that each $\tilde{\pi}_x$ is a homomorphism to check that

$$\begin{aligned}
&\left\| \int_X^{\oplus} \pi'_x(a)d\mu(x) - \int_X^{\oplus} \tilde{\pi}_x(a)d\mu(x) \right\| \\
&\leq \left\| \int_X^{\oplus} \pi'_x(a)d\mu(x) - \int_X^{\oplus} \pi'_x(a_{i_j})d\mu(x) \right\| + \left\| \int_X^{\oplus} \tilde{\pi}_x(a_{i_j})d\mu(x) - \int_X^{\oplus} \tilde{\pi}_x(a)d\mu(x) \right\| \\
&= \|\pi(a) - \pi(a_{i_j})\| + \left\| \int_X^{\oplus} \tilde{\pi}_x(a_{i_j} - a)d\mu(x) \right\| \\
&\leq 2\|a - a_{i_j}\| \to 0.
\end{aligned}$$

Hence $\pi(a) = \int_X^{\oplus} \pi'_x(a)d\mu = \int_X^{\oplus} \tilde{\pi}_x(a)d\mu$ and, as claimed, we can decompose π so that each $\tilde{\pi}_x$ is a homomorphism.

Next, we show that the $\tilde{\pi}_x$ are nondegenerate almost everywhere. This proof is taken from [Arv76, Proposition 4.2.2]. Let a_i be an approximate identity for A and observe that, because π is nondegenerate $\pi(a_i) \to \mathrm{id}$ strongly. Let e_j be a special

fundamental orthogonal sequence of $X * \mathfrak{H}$ and note that $e_j \in \mathcal{L}^2(X * \mathfrak{H}, \mu)$ for all j. Now, we compute

$$\lim_{i\to\infty} \|\pi(a_i)e_j - e_j\|^2 = \lim_{i\to\infty} \int_X \|\tilde{\pi}_x(a_i)e_j(x) - e_j(x)\|^2 d\mu(x) = 0$$

for all j. Since convergence in mean implies a subsequence converges almost everywhere it follows that for each j we can find a subsequence i_{k_j} such that

$$\lim_{k\to\infty} \|\tilde{\pi}_x(a_{i_{k_j}})e_j(x) - e_j(x)\| = 0$$

for all x not in some null set $N_j \subset X$. By induction we can arrange that the $(j+1)$st subsequence is a subsequence of the jth. Then consider the diagonalization $\{a_{i_{k_k}}\}$. As long as $x \notin \bigcup_j N_j$ we have

$$\lim_{k\to\infty} \|\tilde{\pi}_x(a_{i_{k_k}})e_j(x) - e_j(x)\| = 0$$

for all j. Since the $\{e_j(x)\}$ form a basis for $\mathcal{H}(x)$, this is enough to show that $\tilde{\pi}_x$ is nondegenerate almost everywhere.

Now, let I_x be the ideal in A such that $A(x) = A/I_x$. We would like to claim that $I_x \subset \ker \tilde{\pi}_x$ for all $x \in X$. However, we are going to have to deal with more almost everywhere nonsense. Recall that I_x is generated by elements of the form $\phi \cdot a$ where $a \in A$ and $\phi \in C_0(X)$ such that $\phi(x) = 0$. Given $a \in A$, $\phi \in C_0(X)$ and $f \in \mathcal{L}^2(X * \mathfrak{H})$ we have

$$\int_X^{\oplus} \tilde{\pi}_x(\phi \cdot a) d\mu(x) f = \pi(\phi \cdot a) f = T_\phi \pi(a) f = T_\phi \int_X^{\oplus} \tilde{\pi}_x(a) d\mu(x) f$$

where T_ϕ is the diagonal operator associated to ϕ. This implies that there exists a null set $N(\phi, a, f)$ such that

$$\tilde{\pi}_x(\phi \cdot a) f(x) = \phi(x)\tilde{\pi}_x(a) f(x)$$

for all $x \notin N(\phi, a, f)$. However, if we let $\{\phi_i\}$ and $\{a_j\}$ be countable dense subsets of $C_0(X)$ and A, respectively, and e_l a special orthogonal fundamental sequence of $X * \mathfrak{H}$ then we can pick a single null set $N = \bigcup N(\phi_i, a_j, e_l)$ such that given $x \notin N$ we have

$$\tilde{\pi}_x(\phi_i \cdot a_j) e_l(x) = \phi_i(x)\tilde{\pi}_x(a_j) e_l(x)$$

for all i, j, l. Since each $\{e_l(x)\}$ forms an orthogonal basis for $\mathcal{H}(x)$ (plus some zero vectors) it follows that given $x \notin N$ we have $\tilde{\pi}_x(\phi_i \cdot a_j) = \phi_i(x)\tilde{\pi}_x(a_j)$ for all i, j. By continuity we get $\tilde{\pi}_x(\phi \cdot a) = \phi(x)\tilde{\pi}_x(a)$ for all $\phi \in C_0(X)$ and $a \in A$ as long as

$x \notin N$. Thus, for $x \notin N$, we get $I_x \subset \ker \tilde{\pi}_x$ and we can assume that this is true all of the time by setting $\tilde{\pi}_x = 0$ on N. Furthermore, since we have only changed $\{\tilde{\pi}_x(a)\}$ on a null set we still have $\pi(a) = \int_X^\oplus \tilde{\pi}_x(a) d\mu(x)$ for all $a \in A$. Now let π_x be the factorization of $\tilde{\pi}_x$ to $A(x)$ for all $x \in X$. Then, since $\pi_x(a(x)) = \tilde{\pi}_x(a)$ for all x, $\{\pi_x(a(x))\}$ is clearly a bounded Borel field of operators for all $a \in A$ and we have

$$\pi(a) = \int_X^\oplus \tilde{\pi}_x(a) d\mu(x) = \int_X^\oplus \pi_x(a(x)) d\mu(x).$$

This yields the required decomposition of π to the fibres $A(x)$. Furthermore, since the factorization of a nondegenerate representation is nondegenerate, the π_x are nondegenerate almost everywhere.

Finally, suppose ρ_x is also a decomposition of π as in the statement of the proposition. Then it follows from Lemma 3.96 that for each $a \in A$ there exists a μ-null set N_a such that for all $x \notin N_a$ we have $\pi_x(a) = \rho_x(a)$. Let a_i be a countable dense set in A and let $N = \bigcup N_{a_i}$. Then N is still μ-null and it follows from the fact that representations of C^*-algebras are continuous that $\pi_x(a) = \rho_x(a)$ for all $a \in A$ and all $x \notin N$. $\square$

It is a useful fact that, at least in the separable case, every representation of a $C_0(X)$-algebra is equivalent to a $C_0(X)$-linear one. The following theorem is a restriction of [Dix77, Theorem 8.3.2].

Proposition 3.101. *Suppose X is a second countable locally compact Hausdorff space, A is a separable $C_0(X)$-algebra and ρ is a separable representation of A on $\mathcal{H}_\rho$. Then there is an analytic Borel Hilbert bundle $X * \mathfrak{H}$ and a finite measure μ on X such that ρ is unitarily equivalent to a $C_0(X)$-linear representation of A on $L^2(X * \mathfrak{H}, \mu)$.*

Proof. Suppose the $C_0(X)$-action on A is given by Φ_A. Given ρ as above we can extend ρ to the multiplier algebra and obtain a representation $\rho' = \overline{\rho} \circ \Phi_A$ of $C_0(X)$ on $\mathcal{H}_\rho$ which is nondegenerate because Φ_A and ρ are. It is a deep result ([Wil07, Theorem E.14],[Arv76, Pages 54–55]) that ρ' is unitarily equivalent to a representation of the form

$$\tilde{\rho} = (\rho_{\mu_\infty} \otimes 1_{\mathcal{H}_\infty}) \oplus \rho_{\mu_1} \oplus (\rho_{\mu_2} \otimes 1_{\mathcal{H}_2}) \oplus \cdots$$

on

$$(L^2(X_\infty, \mu_\infty) \otimes \mathcal{H}_\infty) \oplus L^2(X_1, \mu_1) \oplus (L^2(X_2, \mu_2) \otimes \mathcal{H}_2) \oplus \cdots$$

where each μ_n is a finite Borel measure on X with μ_n disjoint from μ_m if $n \neq m$, ρ_{μ_n} is the representation of $C_0(X)$ on $L^2(X, \mu_n)$ given by $\rho_{\mu_n}(\phi)h(x) = \phi(x)h(x)$ for all $\phi \in C_0(X)$ and $h \in L^2(X, \mu_n)$, and $\mathcal{H}_n$ is a fixed Hilbert space of dimension $0 \leq n \leq \infty$. Since there is no harm in replacing μ_n by a scalar multiple of μ_n we can

assume without loss of generality that $\mu_n(X) \leq 1/2^n$. Let $X_n = \operatorname{supp} \mu_n$ and observe that X_n is a Borel partition of X. (If we are missing any bits just throw them into X_1 and give them zero measure.) Let $X * \mathfrak{H}$ be the disjoint union $\coprod_{n=0}^{n=\infty} X_n \times \mathcal{H}_n$. It is easy to see that $X * \mathfrak{H}$ is an analytic Borel Hilbert bundle [Wil07, Example F.5]. Now let $\mu = \sum_n \mu_n$ and note that μ is a finite Borel measure on X. It is straightforward to see [Wil07, Corollary F.12] that

$$L^2(X * \mathfrak{H}, \mu) \cong (L^2(X_\infty, \mu_\infty) \otimes \mathcal{H}_\infty) \oplus L^2(X_1, \mu_1) \oplus (L^2(X_2, \mu_2) \otimes \mathcal{H}_2) \oplus \cdots$$

and that the isomorphism is given by sending a section in $L^2(X * \mathfrak{H}, \mu)$ to the sum of the appropriate restrictions to each subfactor. However, once we untangle all of the definitions it follows that this isomorphism intertwines $\tilde{\rho}$ with the representation ρ_μ of $C_0(X)$ on $L^2(X * \mathfrak{H}, \mu)$ given by $\rho_\mu(\phi) = T_\phi$ for all $\phi \in C_0(X)$ where T_ϕ is the diagonal operator associated to ϕ. (Notice that $\rho_{\mu_n}(\phi)$ is exactly the "diagonal" operator associated to ϕ in $L^2(X, \mu_n)$.)

Next, let $U : \mathcal{H}_\rho \to L^2(X * \mathfrak{H}, \mu)$ be the unitary intertwining ρ' and ρ_μ and let $\pi(a) = U\rho(a)U^*$. Then π is unitarily equivalent to ρ by construction and we can compute that

$$\pi(\Phi_A(\phi)a) = U\overline{\rho}(\Phi_A(\phi))U^*U\rho(a)U^* = U\rho'(\phi)U^*\pi(a) = \rho_\mu(\phi)\pi(a) = T_\phi\pi(a).$$

Thus π is $C_0(X)$-linear. $\square$

Remark 3.102. Of course, combined with Proposition 3.99, this implies that *every* separable representation of a separable $C_0(X)$-algebra is unitarily equivalent to a direct integral of representations of the fibres.

At this point we are finally ready to define what it means to be a covariant representation of a groupoid crossed product.

Definition 3.103. Suppose (A, G, α) is a separable groupoid dynamical system. A *covariant representation* $(\mu, G^{(0)} * \mathfrak{H}, \pi, U)$ of (A, G, α) consists of a unitary representation $(\mu, G^{(0)} * \mathfrak{H}, U)$ of G and a $C_0(G^{(0)})$-linear representation $\pi : A \to B(L^2(X * \mathfrak{H}, \mu))$. Furthermore, if $\{\pi_x\}$ is a decomposition of π and ν is the measure induced by μ we require that there exists a ν-null set N such that for all $\gamma \notin N$ we have

$$U_\gamma \pi_{s(\gamma)}(a) = \pi_{r(\gamma)}(\alpha_\gamma(a))U_\gamma \quad \text{for all } a \in A(s(\gamma)). \tag{3.16}$$

Remark 3.104. To conserve notation we will sometimes denote a covariant representation by (π, U) and will understand that the groupoid representation includes a quasi-invariant measure and a Borel Hilbert bundle.

We end this section with an example of a very important class of representations. Unfortunately, the example is fairly technical and requires technology we have not

introduced here. The following attempts to describe a coherent construction but certainly does not include all of the relevant details.[5] One of the things to take away from this example is that dealing with covariant representations directly is messy. We will develop tools in the next section which will allow us to get around these difficulties.

Example 3.105. Suppose (A, G, α) is a separable groupoid dynamical system and π is a separable representation of A. Using Proposition 3.101 we can assume without loss of generality that there exists a Borel Hilbert bundle $X * \mathfrak{H}$ and a finite measure μ such that π is a $C_0(G^{(0)})$-linear representation of A on $L^2(X * \mathfrak{H}, \mu)$. As such there is a decomposition

$$\pi = \int_{G^{(0)}}^{\oplus} \pi_u d\mu(u)$$

coming from Proposition 3.99. Next, let $[\mu]$ be the saturation of μ and $\nu = \int_{G^{(0)}} \lambda^u d\mu$ be the measure induced on G by μ. We use [Wil07, Theorem I.5] to decompose ν with respect to the source map to get measures ν_u on G_u such that

$$\nu(f) = \int_{G^{(0)}} \int_G f(x) d\nu_u(x) d[\mu](u).$$

We denote the image of ν_u under inversion as ν^u. Observe that ν^u is a Radon measure on G^u. Now, use Example 3.84 to form the pull back bundle $s^*(G^{(0)} * \mathfrak{H})$. We define

$$\mathcal{K}(u) = L^2(s^*(G^{(0)} * \mathfrak{H})|_{G^u}, \nu^u)$$

for all $u \in G^{(0)}$. In other words, $\mathcal{K}(u)$ is the L^2-space of maps from G^u into $G^{(0)} * \mathfrak{H}$ such that $h(\gamma) \in \mathcal{H}(s(\gamma))$ for all $\gamma \in G^u$. Since each ν^u is finite, this is a collection of separable Hilbert spaces which we then form into the bundle $G^{(0)} * \mathfrak{K}$. Now, let f_n be a fundamental sequence for $s^*(G^{(0)} * \mathfrak{H})$ and ϕ_m a sequence of point separating functions in $C_c(G)$. We then define a fundamental sequence on $G^{(0)} * \mathfrak{K}$ by

$$g_{n,m}(u)(\gamma) := \phi_m(\gamma) f_n(\gamma).$$

It is straightforward to show that Proposition 3.65 implies that we can use the $g_{n,m}$ to make $G^{(0)} * \mathfrak{K}$ into an analytic Borel Hilbert bundle.

Let Δ be the modular function coming from $[\mu]$. Now, it takes a lot of work, is not at all obvious, and currently there is no decent reference,[6] but we can modify the ν^u on a null set so that $\gamma \cdot \nu^{s(\gamma)} = \Delta(\gamma)\nu^{r(\gamma)}$ for all $\gamma \in G$. This allows us to define a unitary

$$L_\gamma : L^2(s^*(G^{(0)} * \mathfrak{H})|_{G^{s(\gamma)}}, \nu^{s(\gamma)}) \to L^2(s^*(G^{(0)} * \mathfrak{H})|_{G^{r(\gamma)}}, \nu^{r(\gamma)})$$

[5] Many thanks to Jon Brown and Dana Williams for allowing me to use their notes as a reference.

[6] There is a proof of this fact in the personal notes of Dana Williams.

by $L_\gamma h(\eta) = \Delta(\gamma)^{\frac{1}{2}} h(\gamma^{-1}\eta)$ for all $\eta \in G^{r(\gamma)}$. This is just a souped up version of the left regular representation from Example 3.78 and it is not hard to see that $L : G \to \mathrm{Iso}(G^{(0)} * \mathfrak{K})$ such that $L(\gamma) = (r(\gamma), L_\gamma, s(\gamma))$ defines a representation of G. Finally, we define $\tilde{\pi} : A \to L^2(G^{(0)} * \mathfrak{K}, [\mu])$ by

$$(\tilde{\pi}(a)\xi(u))(\gamma) = \pi_{s(\gamma)}(\alpha_\gamma^{-1}(a(u)))\xi(u)(\gamma)$$

for $a \in A$ and $\xi \in L^2(G^{(0)} * \mathfrak{K}, [\mu])$. It takes some work but one can show that $\tilde{\pi}$ is a $C_0(G^{(0)})$-linear representation of A and that its decomposition is given by

$$\tilde{\pi}_u(a)h(\gamma) = \pi_{s(\gamma)}(\alpha_\gamma^{-1}(a))h(\gamma).$$

for $a \in A(u)$ and $h \in \mathcal{K}(u)$. Finally, given $\gamma \in G$, we can observe that

$$\begin{aligned}
(L_\gamma \tilde{\pi}_{s(\gamma)}(a)h)(\eta) &= \Delta(\gamma)^{\frac{1}{2}} \tilde{\pi}_{s(\gamma)}(a)h(\gamma^{-1}\eta) \\
&= \Delta(\gamma)^{\frac{1}{2}} \pi_{s(\eta)}(\alpha_{\gamma^{-1}\eta}^{-1}(a))h(\gamma^{-1}\eta) \\
&= \pi_{s(\eta)}(\alpha_\eta^{-1}(\alpha_\gamma(a)))L_\gamma h(\eta) \\
&= (\tilde{\pi}_{r(\gamma)}(\alpha_\gamma(a))L_\gamma h)(\eta)
\end{aligned}$$

for $a \in A(s(\gamma))$, $h \in \mathcal{K}(s(\gamma))$ and $\eta \in G^{r(\gamma)}$. Thus $([\mu], G^{(0)} * \mathfrak{K}, \tilde{\pi}, L)$ is a covariant representation of (A, G, α) called the *left regular representation.*

3.4 The Groupoid Crossed Product

We start with a couple of useful propositions that are mildly interesting from a measure theoretic point of view. The following proofs were communicated to me by Dana Williams.[7]

Definition 3.106. Suppose that X and Y are locally compact Hausdorff spaces. A family $\{\rho^y\}_{y\in Y}$ of Radon measures on X is called a *Radon family* of measures on X if

$$y \mapsto \int_X f(y)d\rho^y(x) \tag{3.17}$$

is a bounded Borel function supported on a compact set for all $f \in C_c(X)$.

Example 3.107. The most common example of a Radon family of measures will be a Haar system on a groupoid G.

[7]Thanks Dana!

Remark 3.108. Suppose X and Y are locally compact Hausdorff spaces and $\{\rho^y\}$ is a Radon family of measures on X. Then if μ is any Radon measure on Y setting

$$\nu(f) := \int_Y \int_X f(x) d\rho^y(x) d\mu(y) \tag{3.18}$$

for all $f \in C_c(X)$ defines a radon measure ν on X called the *induced measure*. Our goal will be to see that (3.18) extends to a much wider class of functions.

Proposition 3.109. *Suppose X and Y are second countable locally compact Hausdorff spaces and $\{\rho^y\}$ is a Radon family of measures on X. Let μ be a Radon measure on Y and let $\nu = \int \rho^y \mu(u)$ be the induced measure as in Remark 3.108. Given a positive Borel function f we define $\rho(f)$ on Y by*

$$\rho(f)(y) := \int_X f(x) d\rho^y(x) \tag{3.19}$$

Then $\rho(f)$ is a positive extended real valued Borel function on Y and we have

$$\int_X f(x) d\nu(x) = \int_Y \rho(f)(y) d\mu(y). \tag{3.20}$$

Furthermore if f is a ν-integrable Borel function then we define $\rho(f)$ on Y by (3.19) *whenever f is ρ^y-integrable and $\rho(f) = 0$ otherwise. Then in this case $\rho(f)$ is a Borel function on Y and* (3.20) *still holds.*

Remark 3.110. When convenient we will use (3.18) instead of the more formal (3.20).

Remark 3.111. We will use the following notation. Given a locally compact Hausdorff space X we will let $\mathcal{B}(X)$ denote the set of Borel functions on X and $\mathcal{B}_c^b(X)$ denote the set of bounded Borel functions which vanish off a compact set. We will let $\mathcal{B}^+(X)$ denote the positive Borel functions and $\mathcal{B}^{+,e}(X)$ denote the set of positive extended real valued Borel functions.

We will prove Proposition 3.109 via a series of lemmas.

Lemma 3.112. *If $f \in \mathcal{B}_c^b(X)$ then $\rho(f) \in \mathcal{B}_c^b(Y)$. If $f \in \mathcal{B}^+(X)$ then $\rho(f) \in \mathcal{B}^{+,e}(Y)$.*

Proof. Note that $\rho(f)$ is defined if $f \in \mathcal{B}_c^b(X)$ or $f \in \mathcal{B}^+(X)$, although in the latter case the function may take infinite values. If $f \in \mathcal{B}_c^b(X)$ then there is a $g \in C_c^+(X)$ such that $|f| \leq g$. Then

$$|\rho(f)| \leq \rho(|f|) \leq \rho(g).$$

Since $\{\rho^y\}$ is a Radon family, $\rho(g) \in \mathcal{B}_c^b(Y)$, and it follows that $\rho(f)$ is bounded and must vanish off a compact set. Thus if $f \in \mathcal{B}_c^b(X)$ then $\rho(f) \in \mathcal{B}_c^b(Y)$ exactly when $\rho(f)$ is Borel.

Now let $U \subset X$ be a relatively compact open set. Let

$$\mathcal{A} := \{f \in \mathcal{B}^b(U) : \rho(f) \text{ is Borel}\}.$$

Then $C_c(U) \subset \mathcal{A}$. The Dominated Convergence Theorem implies that $\mathcal{A}$ is closed under monotone sequential limits. Then [Ped89, Proposition 6.2.9] implies that $\mathcal{A} = \mathcal{B}^b(U) \subset \mathcal{B}^b_c(X)$. Since this holds for any relatively compact open set we see that $\rho(f) \in \mathcal{B}^b_c(Y)$ for any $f \in \mathcal{B}^b_c(X)$. If $f \in \mathcal{B}^+(X)$ then, since X is σ-compact, we can find $f_n \in \mathcal{B}^b_c(X) \cap \mathcal{B}^+(X)$ such that $f_n \nearrow f$. We then see that $\rho(f) \in \mathcal{B}^{+,e}(Y)$ by the Monotone Convergence Theorem. □

Lemma 3.113. *We get a measure $\bar{\nu}$ on X via*

$$\bar{\nu}(E) := \int_Y \int_X \chi_E(x) d\rho^y(x) d\mu(y)$$

for any Borel set $E \subset X$.

Proof. Since $\rho(\chi_E)$ is in $\mathcal{B}^{+,e}(Y)$ we know $\bar{\nu}$ is defined on all Borel sets E. Clearly $\bar{\nu}(\emptyset) = 0$. To see that $\bar{\nu}$ is countably additive requires only a couple applications of the Monotone Convergence Theorem. □

Lemma 3.114. *For all $f \in \mathcal{B}^+(X)$*

$$\bar{\nu}(f) := \int_X f(x) d\bar{\nu}(x) = \int_Y \int_X f(x) d\rho^y(x) d\mu(y). \tag{3.21}$$

In particular, $\bar{\nu}$ is a Radon measure on X.

Proof. By linearity, (3.21) holds for all nonnegative simple functions. But if $f \in \mathcal{B}^+(X)$ then there are nonnegative simple functions $f_n \nearrow f$. Thus (3.21) holds for all $f \in \mathcal{B}^+(X)$ by the Monotone Convergence Theorem.

Since X is second countable, to see that $\bar{\nu}$ is a Radon measure just requires that we demonstrate that $\bar{\nu}(K) < \infty$ for all compact sets $K \subset X$ [Rud66, Theorem 2.18]. But we can find $g \in C_c^+(X)$ such that $g(x) = 1$ for all $x \in K$. Then $\bar{\nu}(K) \leq \bar{\nu}(g) < \infty$. □

Proof of Proposition 3.109. If $f \in \mathcal{B}^+(X)$ then the assertions about $\rho(f)$ are taken care of by Lemma 3.112. Furthermore, ν and $\bar{\nu}$ are Radon measures that agree on $C_c(X)$. Thus, $\nu = \bar{\nu}$ which implies that (3.18) holds as required.

Now suppose f is ν-integrable. Then f is $\bar{\nu}$-integrable and we can decompose f as the sum of four positive $\bar{\nu}$-integrable functions $f = f_1 - f_2 + if_3 - if_4$. However it now follows that, for each i, $\rho(f_i)(y) < \infty$ for μ-almost all y. Therefore, f is ρ^y integrable and $\rho(f)$ is defined by (3.19) μ-almost everywhere. The fact that (3.18) holds for f now follows from the fact that it holds for each f_i and by linearity. □

The following result is contained in [Ram71, Lemma 5.2] or [Muh, Lemma 4.9]. However, both of those references make use of measured groupoids, so a measured groupoid free proof is provided here for convenience. This result is particularly useful when dealing with the fact that the covariance relation for covariant representations only holds almost everywhere.

Definition 3.115. Suppose G is a groupoid and N is a G-invariant subset of $G^{(0)}$. Then the *restriction* of G to N is defined to be

$$G|_N := r^{-1}(N) = \{\gamma \in G : r(\gamma), s(\gamma) \in N\}.$$

Lemma 3.116. *Suppose G is a second countable locally compact Hausdorff groupoid with Haar system λ and μ is a finite quasi-invariant measure on $G^{(0)}$. Let $\nu = \int_{G^{(0)}} \lambda^u d\mu(u)$ and suppose Σ is a ν-conull subset of G. If Σ is closed under multiplication then there exists a G-invariant μ-conull set $N \subset G^{(0)}$ such that $G|_N$ is conull and $G|_N \subset \Sigma$.*

Proof. Set $\Sigma_1 = \Sigma \cap \Sigma^{-1} = \{\gamma \in G : \gamma, \gamma^{-1} \in \Sigma\}$. Then Σ_1 is a groupoid contained in G (whose unit space may be smaller than $G^{(0)}$) and contains a ν-conull Borel subset B of G. It follows from Proposition 3.109 applied to the characteristic function of $G \setminus B$ that if B is conull with respect to ν then there must exist a μ-conull set U such that $\lambda^u(G^u \setminus B) = 0$ for all $u \in U$. Now, let $N = r(r^{-1}(U) \cap s^{-1}(U))$. Clearly N is a Borel set. It is straightforward to show that $r^{-1}(U)$ is ν-conull and $s^{-1}(U)$ is ν^{-1}-conull. However, ν is equivalent to ν^{-1} so that $r^{-1}(U) \cap s^{-1}(U)$ is ν-conull. It then follows relatively quickly that N is μ-conull and G-invariant. Furthermore, the previous argument also shows that $G|_N$ is conull. We would like to see that $G|_N \subset \Sigma_1$. Suppose $\gamma \in G|_N$ and let $u = r(\gamma)$ and $v = s(\gamma)$. Then $u, v \in U$ so that $G^u \cap B$ and $G^v \cap B$ are λ^u-conull and λ^v-conull, respectively. However, λ is invariant so $\gamma \cdot (G^v \cap B)$ is λ^u-conull. It follows that the intersection $G^u \cap B \cap \gamma \cdot (G^v \cap B)$ is λ^u-conull and therefore nonempty. Thus there exists $\eta \in B \subset \Sigma^1$ such that $r(\eta) = s(\gamma)$ and $\gamma\eta \in B \subset \Sigma^1$. But Σ^1 is a subgroupoid so therefore $\gamma \in \Sigma^1$. □

This leads us, as promised, to the following proposition. This will be extremely useful when dealing with covariant representations because it will allow us to restrict our attention to a conull subgroupoid (what some people call an inessential contraction) on which the covariance relation (3.16) holds.

Proposition 3.117. *Suppose (A, G, α) is a separable groupoid dynamical system and $(\mu, G^{(0)} * \mathfrak{H}, \pi, U)$ is a covariant representation of (A, G, α). Then there exists a G-invariant μ-conull set $N \subset G^{(0)}$ such that* (3.16) *holds on all of the conull subgroupoid $G|_N$.*

Proof. Let Σ be the set of $\gamma \in G$ such that (3.16) holds and let $\nu = \int_{G^{(0)}} \lambda^u d\mu(u)$. By definition Σ is ν-conull. Since U and α are both homomorphisms, it is not hard to see that Σ is closed under multiplication. However the result then follows from Lemma 3.116. □

The following construction is a key aspect of crossed product theory. It is (mostly) developed in [MW08, Section 7] but is so fundamental that it has been reproduced here.

Proposition 3.118. *Suppose (A, G, α) is a separable dynamical system and that $(\mu, G^{(0)} * \mathfrak{H}, \pi, U)$ is a covariant representation. Let $\pi = \int_{G^{(0)}}^{\oplus} \pi_u d\mu$ be a decomposition of π. Then there is an I-norm decreasing, nondegenerate, $*$-representation $\pi \rtimes U$ of $\Gamma_c(G, r^*\mathcal{A})$ on $L^2(G^{(0)} * \mathfrak{H}, \mu)$ given by*

$$\pi \rtimes U(f)h(u) = \int_G \pi_u(f(\gamma))U_\gamma h(s(\gamma))\Delta(\gamma)^{-\frac{1}{2}} d\lambda^u(\gamma) \tag{3.22}$$

for $f \in \Gamma_c(G, r^*\mathcal{A})$, $h \in \mathcal{L}^2(G^{(0)} * \mathfrak{H}, \mu)$, and $u \in G^{(0)}$ where Δ is the modular function from Definition 3.70.

Furthermore, given $h, k \in \mathcal{L}^2(G^{(0)} * \mathfrak{H}, \mu)$ and $f \in \Gamma_c(G, r^*\mathcal{A})$ we have

$$(\pi \rtimes U(f)h, k) = \int_G (\pi_{r(\gamma)}(f(\gamma))U_\gamma h(s(\gamma)), k(r(\gamma)))\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \tag{3.23}$$

where ν is the measure induced by μ.

Proof. Using the quasi-invariance of μ and the Cauchy-Schwartz inequality in $\mathcal{H}(x)$ and $L^2(G, \nu)$ we have, for all $f \in \Gamma_c(G, r^*\mathcal{A})$ and $h, k \in \mathcal{L}^2(G^{(0)} * \mathfrak{H}, \mu)$,

$$\begin{aligned}
&\int_G |(\pi_{r(\gamma)}(f(\gamma))U_\gamma h(s(\gamma)), k(r(\gamma)))|\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \\
&\leq \int_G \|\pi_{r(\gamma)}(f(\gamma))\|\|U_\gamma h(s(\gamma))\|\|k(r(\gamma))\|\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \\
&\leq \int_G \|f(\gamma)\|\|h(s(\gamma))\|\|k(r(\gamma))\|\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \\
&\leq \left(\int_G \|f(\gamma)\|\|h(s(\gamma))\|^2 \Delta(\gamma)^{-1} d\nu(\gamma)\right)^{1/2} \\
&\qquad \cdot \left(\int_G \|f(\gamma)\|\|k(r(\gamma))\|^2 d\nu(\gamma)\right)^{1/2} \\
&\leq \left(\int_{G^{(0)}} \int_G \|f(\gamma)\| d\lambda_u(\gamma)\|h(u)\|^2 d\mu(u)\right)^{1/2}
\end{aligned}$$

$$\cdot \left(\int_{G^{(0)}} \int_G \|f(\gamma)\| d\lambda^u(\gamma) \|k(u)\|^2 d\mu(u) \right)^{1/2}$$
$$\leq \left(\|f\|_I \|h\|^2 \right)^{1/2} \left(\|f\|_I \|k\|^2 \right)^{1/2} = \|f\|_I \|h\| \|k\|.$$

Thus $\gamma \mapsto (\pi_{r(\gamma)}(f(\gamma))U_\gamma h(s(\gamma)), k(r(\gamma)))\Delta(\gamma)^{-\frac{1}{2}}$ is an integrable function, and, using elementary tensors, it is straightforward to see that it is Borel. Equation (3.23) now follows quickly from Proposition 3.109. Let $R = \pi \rtimes U$. It then follows from the above calculation that

$$|(R(f)h|k)| \leq \int_G |(\pi_{r(\gamma)}(f(\gamma))U_\gamma h(s(\gamma)), k(r(\gamma)))| \Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \leq \|f\|_I \|h\| \|k\|.$$

This is enough to show that R is I-norm decreasing.

It is clear that R is linear. We show now that R is multiplicative. Using Proposition 3.117 we can assume that there is a μ-null set N such that the convolution identity holds on all of $G|_N$. If $f, g \in \Gamma_c(G, r^*\mathcal{A})$, $h \in \mathcal{L}^2(G^{(0)} * \mathfrak{H}, \mu)$ and $u \in N$ we have

$$\begin{aligned}
R(f * g)h(u) &= \int_G \pi_u(f * g(\gamma))U_\gamma h(s(\gamma))\Delta(\gamma)^{-\frac{1}{2}} d\lambda^u(\gamma) \\
&= \int_G \int_G \pi_u(f(\eta)\alpha_\eta(g(\eta^{-1}\gamma)))U_\gamma h(s(\gamma))\Delta(\gamma)^{-\frac{1}{2}} d\lambda^u(\eta)\lambda^u(\gamma) \\
&= \int_G \pi_u(f(\eta)) \int_G \pi_u(\alpha_\eta(g(\eta^{-1}\gamma)))U_\gamma h(s(\gamma))\Delta(\gamma)^{-\frac{1}{2}} d\lambda^u(\gamma) d\lambda^u(\eta) \\
&= \int_G \pi_u(f(\eta)) \int_G \pi_u(\alpha_\eta(g(\gamma)))U_{\eta\gamma} h(s(\gamma))\Delta(\eta\gamma)^{-\frac{1}{2}} d\lambda^{s(\eta)}(\gamma) d\lambda^u(\eta).
\end{aligned}$$

Recall that Theorem 3.72 says that we can choose Δ to be a homomorphism. Using the fact that U is a homomorphism, as well as the covariance relation, we get

$$\begin{aligned}
R(f * g)h(u) &= \int_G \pi_u(f(\eta))U_\eta \int_G \pi_u(g(\gamma))U_\gamma h(s(\gamma))\Delta(\gamma)^{-\frac{1}{2}} d\lambda^{s(\eta)}(\gamma)\Delta(\eta)^{-\frac{1}{2}} d\lambda^u(\eta) \\
&= \int_G \pi_u(f(\eta))U_\eta R(g)h(s(\eta))\Delta(\eta)^{-\frac{1}{2}} d\lambda^u(\eta) \\
&= R(f)R(g)h(u).
\end{aligned}$$

Next, we will prove that R preserves the involution. It is straightforward to show that $\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma)$ is invariant under inversion. Using this fact we have, for

appropriate f,h and k,

$$\begin{aligned}
(R(f^*)h,k) &= \int_G (\pi_{r(\gamma)}(f^*(\gamma))U_\gamma h(s(\gamma)), k(r(\gamma)))\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \\
&= \int_G (\pi_{r(\gamma)}(\alpha_\gamma(f(\gamma^{-1})^*))U_\gamma h(s(\gamma)), k(r(\gamma)))\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \\
&= \int_G (\pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\gamma)))^* U_\gamma^* h(r(\gamma)), k(s(\gamma)))\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \\
&= \int_G (h(r(\gamma)), U_\gamma \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\gamma)))k(s(\gamma)))\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \\
&= \int_G (h(r(\gamma)), \pi_{r(\gamma)}(f(\gamma))U_\gamma k(s(\gamma)))\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \\
&= (h, R(f)k).
\end{aligned}$$

It follows that R is a $*$-homomorphism.

The last thing we have to prove is that R is nondegenerate. Suppose $(R(f)h,k) = 0$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$ and $h \in L^2(X * \mathfrak{H}, \mu)$. Let e_l be a special orthogonal fundamental sequence in $L^2(X * \mathfrak{H}, \mu)$ and a_i a countable dense set in A. Since G is second countable it is σ-compact and we can find a countable set of functions $\phi_n \in C_c(G)^+$ such that for all $\gamma \in G$ there exists ϕ_n such that $\phi_n(\gamma) = 1$. Now let $\mathcal{H}_0$ be the countable set of rational linear combinations of e_l and observe that

$$0 = (R(\phi_n \otimes a_i)h, k) = \int_G \phi_n(\gamma)(\pi_{r(\gamma)}(a_i(r(\gamma)))U_\gamma h(s(\gamma)), k(r(\gamma)))\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma)$$

for all n, i and $h \in \mathcal{H}_0$. Since this is a countable family there exists a ν-null set N such that, for all $\gamma \notin N$, we have

$$\phi_n(\gamma)(\pi_{r(\gamma)}(a_i(r(\gamma)))U_\gamma h(s(\gamma)), k(r(\gamma)))$$

for all n, i and $h \in \mathcal{H}_0$. Now $0 = \nu(N) = \int_{G^{(0)}} \lambda^u(N) d\mu(u)$ so that there exists a μ-null set M such that $\lambda^u(N) = 0$ for all $u \notin M$. Furthermore, by making M a little larger if necessary, we can assume that π_u is nondegenerate for all $u \notin M$. Then given $u \notin M$ choose some $\gamma \in G^u$ such that $\gamma \notin N$ and n such that $\phi_n(\gamma) = 1$. Then

$$(\pi_u(a_i(u))U_\gamma h(s(\gamma)), k(u)) = 0 \tag{3.24}$$

for all i and $h \in \mathcal{H}_0$. Now, observing that $\{e_l(s(\gamma))\}$ forms an orthogonal basis for $\mathcal{H}(s(\gamma))$ (plus zero vectors) and that U_γ is a unitary, it follows that $\{U_\gamma h(s(\gamma))\}_{h \in \mathcal{H}_0}$ is dense in $\mathcal{H}(r(\gamma))$. Furthermore, $\{a_i(u)\}$ is dense in $A(u)$ and π_u is nondegenerate so $\{\pi_u(a_i(u))U_\gamma h(s(\gamma))\}$ is dense in $\mathcal{H}(r(\gamma))$. It now follows from (3.24) that $k(u) = 0$.

Since this is true μ-almost everywhere, we have $k = 0$ and that R is nondegenerate. □

This brings us to the most important tool in the theory of groupoid crossed products. The following theorem is a generalization of Renault's "Proposition 4.2" which we will discuss in Section 4.2. It is stated and proved in [MW08]. The reason it's so important is that it allows us to avoid dealing with covariant representations directly.

Theorem 3.119 (Renault's Disintegration Theorem [MW08, Theorem 7.12])**.** *Suppose that $\mathcal{H}_0$ is a dense subspace of a complex Hilbert space $\mathcal{H}$ and that π is a homomorphism from $\Gamma_c(G, r^*\mathcal{A})$ into the algebra of linear operators on $\mathcal{H}_0$ such that*

(a) $\operatorname{span}\{\pi(f)h : f \in \Gamma_c(G, r^*\mathcal{A}), h \in \mathcal{H}_0\}$ *is dense in $\mathcal{H}$,*

(b) *for each $h, k \in \mathcal{H}_0$,*
$$f \mapsto (\pi(f)h, k)$$
is continuous in the inductive limit topology, and

(c) *for each $f \in \Gamma_c(G, r^*\mathcal{A})$ and all $h, k \in \mathcal{H}_0$*
$$(\pi(f)h, k) = (h, \pi(f^*)k).$$

Then each $\pi(f)$ is bounded and extends to a bounded operator $\Pi(f)$ on $\mathcal{H}$ such that Π is an I-norm decreasing $*$-representation of $\Gamma_c(G, r^*\mathcal{A})$. Furthermore, there is a covariant representation $(\mu, G^{(0)} * \mathfrak{H}, \rho, U)$ such that Π is equivalent to the integrated representation $\rho \rtimes U$.

Remark 3.120. It is worth pointing out that at this point we are deeply dependent on separability hypotheses. They were essential in disintegrating representations of $C_0(X)$-algebras and they are also essential to proving the disintegration theorem above. This is still true if we restrict to groupoid algebras, so that separability assumptions will be required in that case as well.

One easy application of the disintegration theorem is the following

Corollary 3.121. *Suppose (A, G, α) is a separable dynamical system and π is a (nondegenerate) $*$-representation of $\Gamma_c(G, r^*\mathcal{A})$ on some Hilbert space $\mathcal{H}$. If π is either I-norm decreasing or continuous in the inductive limit topology then π is equivalent to the integrated form of some covariant representation.*

Proof. If π is I-norm continuous then it is continuous in the inductive limit topology so that it suffices to address the case when π is continuous in the inductive limit topology. We will apply Theorem 3.119 with $\mathcal{H}_0 = \mathcal{H}$. Condition (a) holds since π is

nondegenerate. Condition (c) holds because π is a $*$-homomorphism. Given $f_i \to f$ in the inductive limit topology we know that $\pi(f_i) \to \pi(f)$. However, this implies that $\pi(f_i) \to \pi(f)$ with respect to the weak operator topology and it follows that (b) holds as well. Hence, π is equivalent to the integrated form of some covariant representation and we are done. $\square$

This example presents a different manifestation of the left regular representation that is much easier to deal with. We have developed it here in more generality than is strictly necessary for our purposes. As a result there are some Borel subtleties that make things a bit more confusing.

Example 3.122. Suppose (A, G, α) is a separable dynamical system and π is a nondegenerate representation of A on a separable Hilbert space. Using Proposition 3.101, assume without loss of generality that π is a $C_0(G^{(0)})$-linear representation with decomposition

$$\pi = \int_{G^{(0)}}^{\oplus} \pi_u d\mu(u)$$

on $L^2(G^{(0)} * \mathfrak{H}, \mu)$. Let $\nu_Y^{-1} = \int_{G^{(0)}} \lambda_u d\mu(u)$ be the induced measure and $s^*(G^{(0)} * \mathfrak{H})$ the pull-back bundle as in Example 3.84. We define the *integrated left regular representation* L_μ, usually denoted L, of $\Gamma_c(G, r^*\mathcal{A})$ on $L^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$ by

$$L_\mu(f)h(\gamma) = \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\eta)))h(\eta^{-1}\gamma)d\lambda^{r(\gamma)}(\eta).$$

We will show that L_μ is a nondegenerate, I-norm decreasing, $*$-representation of $\Gamma_c(G, r^*\mathcal{A})$. It is relatively straightforward to see that $L_\mu(f)h \in \mathcal{L}^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$ and that L_μ is linear. Suppose $f, g \in \Gamma_c(G, r^*\mathcal{A})$ and $h, k \in \mathcal{L}^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$. Then we compute

$$\begin{aligned}
C &:= \int_{G^{(0)}} \int_G \int_G |(\pi_u(\alpha_\gamma^{-1}(f(\eta)))h(\eta^{-1}\gamma), k(\gamma))| d\lambda^{r(\gamma)}(\eta) d\lambda_u(\gamma) d\mu(u) \\
&\leq \int_{G^{(0)}} \int_G \int_G \|f(\eta^{-1})\| \|h(\eta\gamma)\| \|k(\gamma)\| d\lambda_{r(\gamma)}(\eta) d\lambda_u(\gamma) d\mu(u) \\
&= \int_{G^{(0)}} \int_G \int_G \|f(\gamma\eta^{-1})\| \|h(\eta)\| \|k(\gamma)\| d\lambda_u(\eta) d\lambda_u(\gamma) d\mu(u).
\end{aligned}$$

Now, we define a measure β on $C_c(G \times G)$ by

$$\beta(g) := \int_G \int_G g(\gamma, \eta) d\lambda_u(\gamma) d\nu^{-1}(\eta) = \int_{G^{(0)}} \int_G \int_G g(\gamma, \eta) d\lambda_u(\gamma) d\lambda_u(\eta) d\mu(u). \quad (3.25)$$

Using the Cauchy-Schwartz inequality with respect to β we get the following inequal-

ity. (Technically, need to know that (3.25) holds for positive Borel functions. However this follows from a straightforward application of Proposition 3.109.)

$$C \le \left(\int_{G^{(0)}} \int_G \int_G \|f(\gamma\eta^{-1})\| \|h(\eta)\|^2 d\lambda_u(\gamma) d\lambda_u(\eta) d\mu(u)\right)^{1/2} \cdot \left(\int_{G^{(0)}} \int_G \int_G \|f(\gamma\eta^{-1})\| \|k(\gamma)\|^2 d\lambda_u(\gamma) d\lambda_u(\eta) d\mu(u)\right)^{1/2} \tag{3.26}$$

Now observe that

$$\begin{aligned}\int_{G^{(0)}} \int_G \int_G &\|f(\gamma\eta^{-1})\| \|h(\eta)\|^2 d\lambda_u(\gamma) d\lambda_u(\eta) d\mu(u) \\ &= \int_G \left(\int_G \|f(\gamma\eta^{-1})\| d\lambda_{s(\eta)}(\gamma)\right) \|h(\eta)\|^2 d\nu^{-1}(\eta) \\ &= \int_G \left(\int_G \|f(\gamma)\| d\lambda_{r(\eta)}(\gamma)\right) \|h(\eta)\|^2 d\nu^{-1}(\eta) \\ &\le \|f\|_I \|h\|^2.\end{aligned}$$

Using a similar computation for the term containing f and k in (3.26) we get

$$\begin{aligned}C &= \int_{G^{(0)}} \int_G \int_G |(\pi_u(\alpha_\gamma^{-1}(f(\eta)))h(\eta^{-1}\gamma), k(\gamma))| d\lambda^{r(\gamma)}(\eta) d\lambda_u(\gamma) d\mu(u) \\ &\le \|f\|_I \|h\| \|k\|.\end{aligned} \tag{3.27}$$

This implies that the function

$$\gamma \mapsto \int_G |(\pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\eta)))h(\eta^{-1}\gamma), k(\gamma))| d\lambda^{r(\gamma)}(\eta)$$

is ν^{-1}-integrable. Applying Proposition 3.109 to ν^{-1} implies that

$$(L_\mu(f)h, k) = \int_{G^{(0)}} \int_G \int_G (\pi_u(\alpha_\gamma^{-1}(f(\eta)))h(\eta^{-1}\gamma), k(\gamma)) d\lambda^{r(\gamma)}(\eta) d\lambda_u(\gamma) d\mu(u) \tag{3.28}$$

Hence (3.27) also implies that $(L_\mu(f)h, k) \le \|f\|_I \|h\| \|k\|$ and therefore L_μ is I-norm decreasing.

Next, we prove that L_μ is multiplicative by computing

$$L_\mu(f * g)h(\gamma) = \int_G \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\zeta)\alpha_\zeta(g(\zeta^{-1}\eta))))h(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\zeta) d\lambda^{r(\gamma)}(\eta)$$

$$
\begin{aligned}
&= \int_G \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\zeta)))\pi_{s(\gamma)}(\alpha_{\zeta^{-1}\gamma}^{-1}(g(\eta)))h(\eta^{-1}\zeta^{-1}\gamma)d\lambda^{s(\zeta)}(\eta)d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\zeta)))(L_\mu(g)h)(\zeta^{-1}\gamma)d\lambda^{r(\gamma)}(\zeta) \\
&= L_\mu(f)L_\mu(g)h(\gamma).
\end{aligned}
$$

We follow up by showing that L_μ preserves involution. Observe that we have to use (3.28) again in order for this computation to hold.

$$
\begin{aligned}
(L_\mu(f^*)h, k) &= \int_{G^{(0)}} \int_G \int_G (\pi_u(\alpha_{\gamma^{-1}\eta}(f(\eta^{-1})^*))h(\eta^{-1}\gamma), k(\gamma))d\lambda^{r(\gamma)}(\eta)d\lambda_u(\gamma)d\mu(u) \\
&= \int_{G^{(0)}} \int_G \int_G (h(\eta^{-1}), \pi_u(\alpha_\eta(f(\eta^{-1}\gamma^{-1})))k(\gamma))d\lambda^u(\eta)d\lambda_u(\gamma)d\mu(u) \\
&= \int_{G^{(0)}} \int_G \int_G (h(\eta), \pi_u(\alpha_\eta^{-1}(f(\eta\gamma)))k(\gamma^{-1}))d\lambda^u(\gamma)d\lambda_u(\eta)d\mu(u) \\
&= \int_G \int_G (h(\eta), \pi_{s(\eta)}(\alpha_\eta^{-1}(f(\gamma)))k(\gamma^{-1}\eta))d\lambda^{r(\eta)}(\gamma)d\nu^{-1}(\eta) \\
&= (h, L_\mu(f)k).
\end{aligned}
$$

This shows that L_μ is a $*$-homomorphism.

Finally, we prove that L_μ is nondegenerate. Suppose $(L_\mu(f)h, k) = 0$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$ and $h \in \mathcal{L}^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$. It follows from (3.28) that

$$
\begin{aligned}
0 &= (L_\mu(f)h, k) \qquad (3.29)\\
&= \int_{G^{(0)}} \int_G \int_G (\pi_u(\alpha_\gamma^{-1}(f(\eta)))h(\eta^{-1}\gamma), k(\gamma))d\lambda^{r(\gamma)}(\eta)d\lambda_u(\gamma)d\mu(u) \\
&= \int_{G^{(0)}} \int_G \int_G (\pi_u(\alpha_\gamma^{-1}(f(\gamma\eta^{-1})))h(\eta), k(\gamma))d\lambda_u(\eta)d\lambda_u(\gamma)d\mu(u).
\end{aligned}
$$

Now, as in the proof of Proposition 3.118, let $\{a_i\}$ be a countable dense set in A and $\{\phi_n\}$ a countable collection of functions in $C_c(G)$ such that for all $\gamma \in G$ there exists n such that $\phi_n(\gamma) = 1$. Let $\{e_l\}$ be a special orthogonal fundamental sequence for $G^{(0)} * \mathfrak{H}$ and let ξ be an arbitrary rational linear combination of the e_l. Observe that $\phi_n \otimes \xi(\gamma) = \phi_n(\gamma)\xi(s(\gamma))$ defines an element in $\mathcal{L}^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$. By applying (3.29) to $\phi_m \otimes a_i$ and $\phi_n \otimes \xi$ for all n,m,i, and ξ we can, by taking a countable union, obtain a single β-null set N such that given $(\gamma, \eta) \notin N$

$$
0 = \phi_m(\gamma\eta^{-1})\phi_n(\eta)(\pi_{s(\gamma)}(\alpha_\gamma^{-1}(a_i(r(\gamma))))\xi(s(\gamma)), k(\gamma)) \qquad (3.30)
$$

for all n, m, i and ξ a rational linear combination of the e_l. Now, it follows from (3.25)

that we can find a ν^{-1}-null set M such that given $\gamma \notin M$ there exists $\eta \in G$ such that $(\eta, \gamma) \notin N$. Pick n and m so that $\phi_m(\gamma\eta^{-1}) = \phi_n(\eta) = 1$. Then (3.30) reduces to

$$0 = (\pi_{s(\gamma)}(\alpha_\gamma^{-1}(a_i(r(\gamma))))\xi(s(\gamma)), k(\gamma)) \tag{3.31}$$

for all a_i, ξ, and $\gamma \notin M$. Next, we can extend M so that $\pi_{r(\gamma)}$ is nondegenerate for all $\gamma \notin M$. Since we have made our choices so that the set $\{\pi_{s(\gamma)}(\alpha_\gamma^{-1}(a_i(r(\gamma))))\xi(s(\gamma))\}$ is dense in $\mathcal{H}(s(\gamma))$, it follows from (3.31) that $k(\gamma) = 0$ for all $\gamma \notin M$. Thus L_μ is nondegenerate.

Note that Corollary 3.121 implies that L_μ must be equivalent to the integrated form of some covariant representation.

Remark 3.123. Readers may have observed that the computations in Example 3.122 are very similar to those done in Proposition 3.118. This is because the representation L_μ is "close" to being the integrated form of the left regular representation from Example 3.105. In fact, L_μ is equivalent to the integrated form of the left regular representation. Let $[\mu]$ be the saturation of μ and $G^{(0)} * \mathfrak{K}$ be as in Example 3.105. Without going into too much detail, the equivalence is implemented by the unitary $U : L^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1}) \to L^2(G^{(0)} * \mathfrak{K}, [\mu])$ where $Uh(u)(\gamma) = h(\gamma)$. We can then compute

$$\begin{aligned}
UL_\mu(f)U^*h(u)(\gamma) &= L_\mu(f)U^*h(\gamma) \\
&= \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\eta)))U^*h(\eta^{-1}\gamma)d\lambda^u(\eta) \\
&= \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\eta)))h(s(\eta))(\eta^{-1}\gamma)\Delta(\eta)^{\frac{1}{2}}\Delta(\eta)^{-\frac{1}{2}}d\lambda^u(\eta) \\
&= \int_G \tilde{\pi}_u(f(\eta))L_\eta h(s(\eta))(\gamma)\Delta^{-\frac{1}{2}}(\eta)d\lambda^u(\eta) \\
&= \tilde{\pi} \rtimes L(f)h(u)(\gamma).
\end{aligned}$$

Of course, we could have saved ourselves some trouble and started by proving that L_μ is equivalent to $\tilde{\pi} \rtimes L$ and then used Proposition 3.118. However, the construction of $(\tilde{\pi}, L)$ was really only outlined in Example 3.105, and it is not entirely obvious that U is a unitary. The fact is that working with the integrated left regular representation L_μ directly is easier. As we have said, one of the great things about Theorem 3.119 is that it frees us up from having to work with covariant representations.

It takes some effort, but we will show that we can separate points in $\Gamma_c(G, r^*\mathcal{A})$ with these integrated regular representations. Unfortunately, we will have to forward reference Lemma 5.10. This isn't a problem, however, since Lemma 5.10 is taken directly from [MW08].

Lemma 3.124. *Suppose (A, G, α) is a separable groupoid dynamical system. Then the (integrated forms of) covariant representations separate points of $\Gamma_c(G, r^*\mathcal{A})$.*

Proof. Clearly it suffices to show that if $f_0 \neq 0$ in $\Gamma_c(G, r^*\mathcal{A})$ then there exists a covariant representation such that $\pi \rtimes U(f_0) \neq 0$. However, once we consider Corollary 3.121 it clearly suffices to show that there exists a (nondegenerate) I-norm decreasing representation L such that $L(f_0) \neq 0$. Find $\gamma_0 \in G$ such that $f_0(\gamma_0) \neq 0$. Set $u = s(\gamma_0)$ and let ρ be a faithful representation of $A(u)$ on $\mathcal{H}$. It is easy to see that the lift of ρ to A, denoted π, is $C_0(G^{(0)})$-linear and that the decomposition is given on the trivial bundle $G^{(0)} \times \mathcal{H}$ by

$$\pi = \int_G \pi_v d\delta_u(v)$$

where $\pi_u = \rho$ and $\pi_v = 0$ for all $v \neq u$. Furthermore, observe that the induced measure ν^{-1} is exactly λ_u in this case. Let us form the integrated left regular representation as in Example 3.122. After sorting through all the definitions we find we have a representation L_{δ_u}, denoted L, of $\Gamma_c(G, r^*\mathcal{A})$ on $L^2(G_u, \mathcal{H}, \lambda_u)$ given by

$$L(f)h(\gamma) = \int_G \rho(\alpha_\gamma^{-1}(f(\eta)))h(\eta^{-1}\gamma)d\lambda^{r(\gamma)}(\eta).^{8} \tag{3.32}$$

Let e_i be an orthonormal basis for $\mathcal{H}$ and for each $g \in \Gamma_c(G, r^*\mathcal{A})$ and $\gamma \in G_u$ define

$$\Phi_i(g)(\gamma) = \rho(\alpha_\gamma^{-1}(g(\gamma)))e_i$$

Then $\Phi_i(g) \in C_c(G_u, \mathcal{H})$ and $\Phi_i : \Gamma_c(G, r^*\mathcal{A}) \to L^2(G_u, \mathcal{H}, \lambda_u)$. What's more, suppose $\Phi_i(g) = 0$ for all i. Then $\rho(\alpha_\gamma^{-1}(g(\gamma)))e_i = 0$ almost everywhere for all i. Thus, we can find a single λ_u-null set N such that $\rho(\alpha_\gamma^{-1}(g(\gamma))) = 0$ for all $\gamma \notin N$. Since ρ is faithful this implies that $g(\gamma) = 0$ for all $\gamma \notin N$. Using the fact that g is continuous and $\operatorname{supp} \lambda_u = G_u$ we conclude that $g(\gamma) = 0$ for all $\gamma \in G_u$.

Next, given $g \in \Gamma_c(G, r^*\mathcal{A})$ we calculate

$$\begin{aligned}
L(f)\Phi_i(g)(\gamma) &= \int_G \rho(\alpha_\gamma^{-1}(f(\eta)))\rho(\alpha_{\eta^{-1}\gamma}^{-1}(g(\eta^{-1}\gamma)))e_i d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \rho(\alpha_\gamma^{-1}(f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))))e_i d\lambda^{r(\gamma)}(\eta) \\
&= \rho(\alpha_\gamma^{-1}(f * g(\gamma)))e_i = \Phi_i(f * g)(\gamma).
\end{aligned}$$

Now suppose, to the contrary, that $L(f_0) = 0$. It follows from Lemma 5.10 that there exists a left approximate identity $\{g_j\}$ in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the inductive limit

[8] Notice that most of the measure theoretic difficulties of Example 3.122 disappear because $\nu^{-1} = \lambda_u$ and $\beta = \lambda_u \times \lambda_u$.

topology. By replacing g_j with g_j^* we may assume that $\{g_j\}$ is a right approximate identity. Thus, $f_0 * g_j \to f_0$ with respect to the inductive limit topology. However, we have $L(f_0)\Phi_i(g_j) = \Phi_i(f_0 * g_j) = 0$ for all i and j. It follows from the previous paragraph that $f_0 * g_j = 0$ on G_u. Since $f_0 * g_j \to f_0$ uniformly, we must have $f_0(\gamma) = 0$ for all $\gamma \in G_u$, but this is a contradiction since $f_0(\gamma_0) \neq 0$. Thus $L(f_0) \neq 0$ and we are done. $\square$

Remark 3.125. We can actually make some considerable upgrades to Lemma 3.124. In particular, we can identify a class of representations of A for which the integrated left regular representation is faithful on $\Gamma_c(G, r^*\mathcal{A})$. Suppose π is a faithful $C_0(G^{(0)})$-linear representation of A with decomposition

$$\pi = \int_{G^{(0)}}^{\oplus} \pi_u d\mu(u)$$

on $L^2(X * \mathfrak{H}, \mu)$ such that π_u is faithful for almost all u. Observe that this happens automatically if A has Hausdorff spectrum.[9] It is straightforward to show that for π to be faithful μ must have full support. It then follows quickly that the induced measure ν^{-1} has full support as well. Let e_i be a special orthogonal fundamental sequence for $X * \mathfrak{H}$ and given $g \in \Gamma_c(G, r^*\mathcal{A})$ define

$$\Phi_i(g)(\gamma) = \pi_{s(\gamma)}(\alpha_\gamma^{-1}(g(\gamma)))e_i(s(\gamma)).$$

It is straightforward to show that $\Phi_i(g)$ is a bounded Borel function supported on a compact set so that for each i we have $\Phi_i : \Gamma_c(G, r^*\mathcal{A}) \to L^2(s^*(X * \mathfrak{H}), \nu^{-1})$. Suppose $\Phi_i(g) = 0$ for all i. Using the usual trick, we can find a single ν^{-1}-null set N such that

$$\pi_{s(\gamma)}(\alpha_\gamma^{-1}(g(\gamma)))e_i(s(\gamma)) = 0$$

for all i given $\gamma \notin N$. Since e_i is a special orthogonal fundamental sequence, this implies that $\pi_{s(\gamma)}(\alpha_\gamma^{-1}(g(\gamma))) = 0$ for all $\gamma \notin N$. We may thicken N a bit and assume that $\pi_{s(\gamma)}$ is faithful for all $\gamma \notin N$ and hence $g(\gamma) = 0$ for all $\gamma \notin N$. Since g is continuous and $\operatorname{supp} \nu^{-1} = G$ this implies that $g = 0$.

Now suppose $f \in \Gamma_c(G, r^*\mathcal{A})$ is such that $L_\mu(f) = 0$. Then, just as in the previous lemma, it is a simple matter to prove that $0 = L_\mu(f)\Phi_i(g) = \Phi_i(f * g)$ for all $g \in \Gamma_c(G, r^*\mathcal{A})$. Use Lemma 5.10 to find a *right* approximate identity $\{g_j\}$ with respect to the inductive limit topology. Then $\Phi_i(f * g_j) = 0$ for all i and j. By the above paragraph this implies that $f * g_j = 0$ for all j. Since $f * g_j \to f$ uniformly this implies $f = 0$. Thus L_μ is faithful on $\Gamma_c(G, r^*\mathcal{A})$. This is related to [Ren80,

[9] It is not clear if many such representations exist in the non-Hausdorff case. There seems to be no reason why the decomposition of a faithful representation should be faithful almost everywhere.

Proposition 1.11], which states that the left regular representation is faithful in the scalar case as long as ν^{-1} has full support.

We are, at long last, ready to define the groupoid crossed product. This definition is slightly different than the definition given in [MW08] but, as will see, because of the Disintegration Theorem the universal norm can be obtained in any number of ways.

Definition 3.126. Suppose (A, G, α) is a separable groupoid dynamical system. We define the *universal norm* on $\Gamma_c(G, r^*\mathcal{A})$ by

$$\|f\| := \sup\{\|\pi \rtimes U(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\}. \quad (3.33)$$

The completion of $\Gamma_c(G, r^*\mathcal{A})$ with respect to this norm is a C^*-algebra called the *groupoid crossed product* of A by G and denoted $A \rtimes_\alpha G$.

Remark 3.127. To avoid clutter we will denote $A \rtimes_\alpha G$ by $A \rtimes G$ whenever the chance of confusion is near zero.

Remark 3.128. There is also the notion of a reduced groupoid crossed product which mimics the group case. Instead of taking the universal norm to be the supremum over all covariant representations, we take it to be the supremum over the left regular representations, either those in Example 3.105 or Example 3.122. While reduced crossed products are interesting, they are less well studied and we will not deal with them here. Those who would like to know more are referenced to [Bro09].

Let us verify the claims made in Definition 3.126 and begin our exploration of the crossed product.

Proposition 3.129. *Suppose (A, G, α) is a separable dynamical system. Then the universal norm is a norm on $\Gamma_c(G, r^*\mathcal{A})$ and is dominated by the I-norm. Furthermore, the completion $A \rtimes_\alpha G$ is a separable C^*-algebra.*

Remark 3.130. It follows that convergence with respect to the I-norm is stronger than convergence with respect to the universal norm. Furthermore, we have from Proposition 3.57 that convergence with respect to the inductive limit topology is stronger than convergence with respect to the I-norm, and therefore with respect to the universal norm as well. We will use this fact often. For example, we can extend Corollary 3.121 and conclude that any representation of $A \rtimes G$ is equivalent to the integrated form of a covariant representation.

Proof. Since the integrated form of a covariant representation is I-norm decreasing, it is clear that $\|f\| \le \|f\|_I$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$. Furthermore, given $f, g \in \Gamma_c(G, r^*\mathcal{A})$ we have

$$\|\pi \rtimes U(f + g)\| \le \|\pi \rtimes U(f)\| + \|\pi \rtimes U(g)\| \le \|f\| + \|g\|$$

and it follows that $\|f+g\| \leq \|f\|+\|g\|$. The fact that $\|\pi \rtimes U(\lambda f)\| = |\lambda|\|\pi \rtimes U(f)\|$ implies that $\|\lambda f\| = |\lambda|\|f\|$ for $\lambda \in \mathbb{C}$. All that remains to show that $\|\cdot\|$ is a norm is to prove that $\|f\| \neq 0$ if $f \neq 0$. However, this follows immediately from Lemma 3.124.

Moving on, it is straightforward to show that $\|f * g\| \leq \|f\|\|g\|$ for $f, g \in \Gamma_c(G, r^*\mathcal{A})$, and since $A \rtimes_\alpha G$ is the completion, it follows that the norm is submultiplicative on the entire crossed product. Similarly we find that $\|f^*\| = \|f\|$ and $\|f^* f\| = \|f\|^2$ for all $f \in A \rtimes_\alpha G$. It follows that $A \rtimes_\alpha G$ is a C^*-algebra. The last thing we need to do is show that $A \rtimes_\alpha G$ is separable. Let $\{a_i\}$ be a countable dense set in A. Since G is second countable we can find a countable set $\{\phi_j\}$ which is dense in $C_c(G)$ with respect to the inductive limit topology. Now, given $\phi \otimes a \in C_c(G) \odot A$ suppose $a_{i_k} \to a$ and $\phi_{j_k} \to \phi$ with respect to the inductive limit topology. Then

$$\|\phi_{j_k}(\gamma)a_{i_k}(r(\gamma)) - \phi(\gamma)a(r(\gamma))\| \leq \|\phi_{j_k} - \phi\|_\infty \|a_{i_k}\| - \|\phi\|_\infty \|a_{i_k} - a\|$$

Since $\{\|a_{i_k}\|\}$ is bounded, this shows that $\phi_{i_k} \otimes a_{i_k} \to \phi \otimes a$ uniformly. Furthermore, since the supports of the ϕ_{j_k} are eventually contained in a fixed compact set, the same is true for $\phi_{j_k} \otimes a_{i_k}$. It follows quickly that the countable set D of rational sums of elements of the form $\phi_j \otimes a_i$ is dense in $C_c(G) \odot A$ with respect to the inductive limit topology, and hence with respect to the I-norm. We conclude from Corollary 3.45 that D is dense in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the I-norm. Since the universal norm is bounded by the I-norm, this enough to show that $A \rtimes G$ is separable. □

The following identities will be quite useful when dealing with crossed products. Both are immediate results of the Disintegration Theorem.

Proposition 3.131. *Suppose (A, G, α) is a separable groupoid dynamical system. Then the universal norm on $\Gamma_c(G, r^*\mathcal{A})$ is also given by*

$$\|f\| = \sup\left\{ \|\pi(f)\| : \begin{array}{l} \pi \textit{ is a (possibly degenerate) } I\textit{-norm decreasing} \\ *\textit{-representation of } \Gamma_c(G, r^*\mathcal{A}) \end{array} \right\}, \tag{3.34}$$

$$\|f\| = \sup\left\{ \|\pi(f)\| : \begin{array}{l} \pi \textit{ is a (possibly degenerate) } *\textit{-representation of } \Gamma_c(G, r^*\mathcal{A}) \\ \textit{which is continuous in the inductive limit topology} \end{array} \right\}. \tag{3.35}$$

It follows that any (possibly degenerate) $*$-representation of $\Gamma_c(G, r^*\mathcal{A})$ which is either I-norm decreasing or continuous with respect to the inductive limit topology is bounded with respect to the universal norm and extends to a representation of $A \rtimes_\alpha G$.

Proof. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be defined by (3.34) and (3.35) respectively. Since every I-norm decreasing representation is continuous with respect to the inductive limit topology, we have $\|\cdot\|_1 \leq \|\cdot\|_2$. Furthermore, since each covariant representation

is I-norm decreasing, we have $\|\cdot\| \leq \|\cdot\|_1$. Now suppose $f \in \Gamma_c(G, r^*\mathcal{A})$ and π is a $*$-representation which is continuous with respect to the inductive limit topology. Let π_{ess} be the restriction to its essential subspace $\mathcal{H}_{\mathrm{ess}} = \overline{\mathrm{span}}\{\pi(f)h : f \in \Gamma_c(G, r^*\mathcal{A}), h \in \mathcal{H}\}$. Let U be the unitary map from $\mathcal{H}$ onto $\mathcal{H}_{\mathrm{ess}} \oplus \mathcal{H}_{\mathrm{ess}}^{\perp}$. Then given $(h,k) \in \mathcal{H}_{\mathrm{ess}} \oplus \mathcal{H}_{\mathrm{ess}}^{\perp}$ we have

$$U\pi(f)U^*(h,k) = U(\pi(f)h + \pi(f)k) = (\pi_{\mathrm{ess}}(f)h + \pi_{\mathrm{ess}}(f)k, 0).$$

However, given $l \in \mathcal{H}$ observe that

$$(\pi(f)k, l) = (k, \pi(f^*)l) = 0$$

since $k \in \mathcal{H}_{\mathrm{ess}}^{\perp}$. It follows that $\pi(f)k = 0$ so that

$$U\pi(f)U^*(h,k) = (\pi_{\mathrm{ess}}(f)h, 0).$$

It follows that π is unitarily equivalent to the representation $\pi_{\mathrm{ess}} \oplus 0$. Hence $\|\pi(f)\| = \|\pi_{\mathrm{ess}} \oplus 0(f)\| = \|\pi_{\mathrm{ess}}(f)\|$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$. Furthermore, it is straightforward to show that, by construction, π_{ess} is a nondegenerate $*$-representation of $\Gamma_c(G, r^*\mathcal{A})$ which is continuous with respect to the inductive limit topology. It follows from Corollary 3.121 that π is equivalent to the integrated form of a covariant representation (ρ, U). Therefore

$$\|\pi(f)\| = \|\pi_{\mathrm{ess}}(f)\| = \|\rho \rtimes U(f)\| \leq \|f\|.$$

Since π was generic, it follows that $\|\cdot\|_2 \leq \|\cdot\|$ and we have demonstrated (3.34) and (3.35). The second half of the proposition is clear from the first half. □

Remark 3.132. This shows that $A \rtimes_\alpha G$ is the "universal enveloping algebra" of $\Gamma_c(G, r^*\mathcal{A})$ with respect to the I-norm, bringing us in line with [MW08].

We end this section with a proposition that we will use liberally. It also hints at the author's general philosophy for working with crossed products, which is "stick to the inductive limit topology whenever possible."

Proposition 3.133. *Suppose (A, G, α) is a separable groupoid dynamical system, D is a C^*-algebra, and that $\Phi : \Gamma_c(G, r^*\mathcal{A}) \to D$ is a $*$-homomorphism such that Φ is either continuous with respect to the inductive limit topology or I-norm decreasing. Then Φ is norm decreasing with respect to the universal norm and extends to a homomorphism of $A \rtimes_\alpha G$ into D.*

Proof. Clearly we can restrict to the case where Φ is continuous with respect to the inductive limit topology. Suppose R is a faithful representation of D. It follows that $R \circ \Phi$ is continuous with respect to the inductive limit topology. Therefore (3.35)

implies that $\|\Phi(f)\| = \|R(\Phi(f))\| \leq \|f\|$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$. Thus Φ is norm decreasing. The remainder of the proposition is clear. $\square$

We usually apply this proposition via the following corollary.

Corollary 3.134. *Suppose (A, G, α) and (B, H, β) are separable groupoid dynamical systems and that $\Phi : \Gamma_c(G, r^*\mathcal{A}) \to \Gamma_c(H, r^*\mathcal{B})$ is a $*$-homomorphism. If $\Phi(f_i) \to \Phi(f)$ with respect to the inductive limit topology whenever $f_i \to f$ with respect to the inductive limit topology then Φ extends to a $*$-homomorphism from $A \rtimes_\alpha G$ into $B \rtimes_\beta H$.*

Proof. This follows immediately from Proposition 3.133 and the fact that convergence with respect to the inductive limit topology is stronger than convergence with respect to the universal norm. $\square$

Chapter 4

Special Cases

In this chapter we show how groupoid crossed products generalize group crossed products in Section 4.1, groupoid C^*-algebras in Section 4.2 and transformation group algebras in Section 4.4. In addition, we will deal with the special case where the groupoid is actually a group bundle in Section 4.3, and when the action arises from a continuous G-space in Section 4.4. However, one thing this chapter, and indeed the whole theory, lacks is natural examples of groupoid dynamical systems which do not come from one of these special cases. The material in this section is essential for two reasons. First, it provides a testing ground for the theory developed in the last chapter and many of the arguments we will use later can be found here in simpler form. Second, the connections made in this chapter provide a paradigm for transporting existing theory to the groupoid case.

4.1 Group Crossed Products

This section deals with group crossed products as defined and developed in [Wil07]. In particular it is assumed that the reader is familiar with at least [Wil07, Chapter 1].

Groupoid crossed products are an extension of group crossed products in two different ways. Both are important. The first is the situation where the groupoid G is a actually group. Start by observing that in this case the unit space of G is a single point $\{e\}$ and that any C^*-algebra A can be viewed as a $C_0(\{e\})$-algebra. In this case A only has one fibre, the bundle associated to A is just A, and the axioms of a dynamical system are reduced to the following

Definition 4.1. Suppose G is a locally compact Hausdorff group and A is a C^*-algebra. An action α of G on A consists of a family of maps $\{\alpha_s\}_{s\in G}$ such that

(a) α_s is an automorphism of A,

(b) $\alpha_{st} = \alpha_s \circ \alpha_t$ for all $s, t \in G$, and

(c) $s \cdot a := \alpha_s(a)$ defines a continuous action of G on A.

Of course, this can be simplified a great deal.

Proposition 4.2. *Suppose G is a locally compact Hausdorff group and A is a C^*-algebra. Then α is an action of G on A if and only if $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism where $\mathrm{Aut}(A)$ is equipped with the topology of pointwise convergence.*

Proof. Suppose α is an action of G on A. It is clear that $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism. Suppose α is a continuous homomorphism. Obviously the first two conditions of Definition 4.1 are satisfied. Now if $s_i \to s$ and $a_i \to a$ then

$$\|\alpha_{s_i}(a_i) - \alpha_s(a)\| \leq \|\alpha_{s_i}(a_i - a)\| + \|\alpha_{s_i}(a) - \alpha_s(a)\| \leq \|a_i - a\| + \|\alpha_{s_i}(a) - \alpha_s(a)\|.$$

Since α is continuous into the topology of pointwise convergence it follows that α is an action of G on A. □

This shows that the restriction of Definition 3.46 to groups is exactly the classical definition of a group action on a C^*-algebra as given in [Wil07, Definition 2.6].

Remark 4.3. Let us consider covariant representations. First recall that any group G has a Haar system given by the Haar measure on G. Furthermore, any measure on $\{e\}$ is trivially quasi-invariant and the modular function is always given by the classical modular function of G with respect to its Haar measure. It is also worth noting that the modular function of a group is always *continuous*. Now, according to Definition 3.76 a (Borel) representation of G is just a Borel homomorphism from G into the unitary operators on some separable Hilbert space equipped with the Borel structure coming from the strong topology. However it follows from [Wil07, Theorem D.3] that in this case U is actually (strongly) continuous. Any representation π of A is trivially $C_0(\{e\})$-linear and, since there is only one fibre, π is its own decomposition. Thus, in the group case, a covariant representation of (A, G, α) is given by a representation π of A and a representation U of G such that

$$U_s \pi(a) = \pi(\alpha_s(a)) U_s \tag{4.1}$$

for almost all $s \in G$. However, U is continuous so that (4.1) holds for all $s \in G$. Thus a covariant representation of (A, G, α) in the groupoid sense is exactly a covariant representation of (A, G, α) as defined in [Wil07, Definition 2.10].

It seems reasonable that, since the covariant representations are the same, then the crossed products of A by G as a group and as a groupoid must be the same. Unfortunately there is the problem of the modular function. First, observe that

$r^*A = G \times A$ and that $\Gamma_c(G, r^*A)$ can be identified with $C_c(G, A)$. Recall from Proposition 3.54 that we defined the operations on $C_c(G, A)$ to be given by

$$f * g(s) = \int_G f(t)\alpha_t(g(t^{-1}s))d\lambda(s), \quad \text{and} \tag{4.2}$$

$$f^*(s) = \alpha_s(f(s^{-1})^*) \tag{4.3}$$

where λ is the Haar measure on G. In [Wil07, Section 2.3] the convolution operation on $C_c(G, A)$ is also defined by (4.3). *However*, the involution operation is defined by

$$f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1})^*). \tag{4.4}$$

Notice that this definition of involution only works in the group case because for groupoids the modular function depends on the choice of the quasi-invariant measure, while for groups the modular function only depends on the Haar measure. This is compensated by the fact that the integrated form of a covariant representation (π, U) as a groupoid dynamical system is given by

$$\pi \rtimes U(f) = \int_G \pi(f(s))U_s\Delta(s)^{-\frac{1}{2}}d\lambda(s)$$

while the integrated form as a group dynamical system is defined in [Wil07, Proposition 2.23] to be

$$\pi \rtimes' U(f) = \int_G \pi(f(s))U_s d\lambda(s).$$

Of course, when we refer to a group crossed product we will use (4.3) and the universal norm coming from Definition 3.126. However, it is important to see that this group crossed product is naturally isomorphic to the one defined in [Wil07, Section 2.3], and to sort out the differences described above.

Proposition 4.4. *Suppose (A, G, α) is a separable dynamical system and that G is a group. Let $A \rtimes_\alpha G$ be the crossed product as defined in Definition 3.126. Let $A \rtimes'_\alpha G$ be the crossed product as defined in [Wil07, Lemma 2.27]. Then the map $\Phi : C_c(G, A) \to C_c(G, A)$ given by $\Phi(f)(s) = \Delta(s)^{-\frac{1}{2}}f(s)$ extends to a $*$-isomorphism of $A \rtimes_\alpha G$ and $A \rtimes'_\alpha G$.*

Remark 4.5. The point of Proposition 4.4 is that for a group dynamical system the modular function can be either included in the formula for involution, or in the formula for the integrated form of a covariant representation. Either way we end up with isomorphic crossed products. We will view group crossed products as special cases of groupoid crossed products.

Proof. Let (A, G) and Φ be as above. Let C denote $C_c(G, A)$ given the operations coming from (4.2) and (4.3) so that $A \rtimes_\alpha G$ is the completion of C. Let D denote $C_c(G, A)$ given the operations (4.2) and (4.4) so that $A \rtimes'_\alpha G$ is the completion of D. Let us calculate

$$\begin{aligned}\Phi(f) * \Phi(g)(s) &= \int_G \Phi(f)(t)\alpha_t(\Phi(g)(t^{-1}s))d\lambda(t) \\ &= \int_G \Delta(t)^{-\frac{1}{2}}\Delta(t^{-1}s)^{-\frac{1}{2}} f(t)\alpha_t(g(t^{-1}s))d\lambda(t) \\ &= \Delta(s)^{-\frac{1}{2}} f * g(s) = \Phi(f * g)(s).\end{aligned}$$

We also have

$$\begin{aligned}\Phi(f)^*(s) &= \Delta(s^{-1})\alpha_s(\Phi(f)(s^{-1})^*) = \Delta(s^{-1})\Delta(s^{-1})^{-\frac{1}{2}}\alpha_s(f(s^{-1})^*) \\ &= \Delta(s)^{-\frac{1}{2}} f^*(s) = \Phi(f^*)(s).\end{aligned}$$

Thus Φ is a $*$-homomorphism. Furthermore, it is clear that Φ is a bijection with inverse given by $\Phi^{-1}(f)(s) = \Delta(s)^{\frac{1}{2}} f(s)$.

Remark 4.6. We would like to see that Φ is bounded and as such extends to the entire crossed product. A valid approach to this problem would be to use the fact that the sets of covariant representations of (A, G, α) as both a group and a groupoid dynamical system are the same and to show that Φ intertwines the two different forms of the integrated representation. Since the universal norms of $A \rtimes G$ and $A \rtimes' G$ are both given by supremums over the covariant representations, the result would follow. However, it is easier, and instructive, to use Proposition 3.131.

Suppose $f_i \to f$ with respect to the inductive limit topology so that $f_i \to f$ uniformly and $\operatorname{supp} f_i \subset K$ eventually, for a fixed compact set K. Since Δ is continuous in the group case we can find M such that $\Delta(s)^{-\frac{1}{2}} < M$ for all $s \in K$. Then we have

$$\|\Phi(f_i)(s) - \Phi(f)(s)\| = \Delta(s)^{-\frac{1}{2}}\|f_i(s) - f(s)\| \leq M\|f_i - f\|_\infty$$

and it follows quickly that $\Phi(f_i) \to \Phi(f)$ with respect to the inductive limit topology. Suppose π is faithful representation of $A \rtimes' G$. We know from [Wil07, Corollary 2.46] that

$$\|f\|_{A \rtimes' G} = \sup\{\|\pi(f)\| : \pi \text{ is continuous in the inductive limit topology}\}. \tag{4.5}$$

It follows that π is continuous in the inductive limit topology so that the composition $\pi \circ \Phi$ is continuous in the inductive limit topology. It is clearly a $*$-homomorphism.

Hence Proposition 3.131 and (3.35) imply that

$$\|\Phi(f)\| = \|\pi(\Phi(f))\| \leq \|f\|.$$

Thus Φ is bounded and extends to $A \rtimes G$. We can use exactly the same argument in reverse to see that Φ^{-1} is bounded and extends to $A \rtimes' G$. Since the extensions of Φ and Φ^{-1} are inverses on a dense set they are inverses everywhere and Φ extends to an isomorphism. $\square$

Now, Proposition 4.4 is theoretically important but it is not the "right" way to view groupoid crossed products as a generalization of group crossed products. The following construction is much more useful in that regard, as we will see in Sections 5.4 and 6.2.

Example 4.7. Suppose A is a separable C^*-algebra with Hausdorff spectrum $\widehat{A}$ and recall from Example 3.24 that we can view A as a $C_0(\widehat{A})$-algebra with fibres $A(\pi) = A/\ker\pi$. Let $\mathcal{A}$ be the corresponding upper-semicontinuous bundle. Suppose α is an action of a second countable locally compact Hausdorff group H on A. Recall from [Wil07, Lemma 2.8] that there is a jointly continuous action of G on $\widehat{A}$ given by $s \cdot \pi = \pi \circ \alpha_s^{-1}$ and let G be the corresponding transformation groupoid $G = H \ltimes \widehat{A}$. Observe that the unit space of G can be identified with $\widehat{A}$ as in Example 1.16.

Our goal is to describe an action of G on A. Given $(s,\pi) \in G$ define $\beta_{(s,\pi)} : A(s^{-1}\cdot\pi) \to A(\pi)$ by

$$\beta_{(s,\pi)}(a(s^{-1}\cdot\pi)) = \alpha_s(a)(\pi). \tag{4.6}$$

We would like to show that $\beta_{(s,\pi)}$ is a well defined isomorphism. First, suppose $a, b \in A$ such that $a(s^{-1}\cdot\pi) = b(s^{-1}\cdot\pi)$. However, this just says that

$$0 = s^{-1}\cdot\pi(a-b) = \pi(\alpha_s(a-b))$$

and hence $\alpha_s(a)(\pi) = \alpha_s(b)(\pi)$. Thus $\beta_{(s,\pi)}$ is well defined. Next, since each α_s is a $*$-homomorphism it is straightforward to show that $\beta_{(s,\pi)}$ is as well. Given $(e,\pi) \in G$ we have

$$\beta_{(e,\pi)}(a(\pi)) = \alpha_e(a)(\pi) = a(\pi)$$

so that $\beta_{(e,\pi)} = \mathrm{id}$. Furthermore, given $(s,\pi), (t, s^{-1}\cdot\pi) \in G$ we have

$$\begin{aligned}\beta_{(s,\pi)} \circ \beta_{(t,s^{-1}\cdot\pi)}(a(t^{-1}s^{-1}\cdot\pi)) &= \beta_{(s,\pi)}(\alpha_t(a)(s^{-1}\cdot\pi)) = \alpha_s(\alpha_t(a))(\pi) \\ &= \alpha_{st}(a)(\pi) = \beta_{(st,\pi)}(a((st)^{-1}\cdot\pi)) \\ &= \beta_{(s,\pi)(t,s^{-1}\cdot\pi)}(a(t^{-1}s^{-1}\cdot\pi)).\end{aligned}$$

It follows that $\beta_{(s^{-1},s^{-1}\cdot\pi)}$ is an inverse to $\beta_{(s,\pi)}$ and that each $\beta_{(s,\pi)}$ is an isomorphism. Furthermore, a byproduct of the above computation is that we have shown β is a

homomorphism. At this point, all we need to do to show that β is an action of G on A is show that it is continuous.

Suppose $(s_i, \pi_i) \to (s, \pi)$ and that $a_i \to a$ in $\mathcal{A}$ such that $a_i \in A(s_i^{-1} \cdot \pi_i)$ for all i and $a \in A(s^{-1} \cdot \pi)$. Choose $b \in A$ so that $b(s^{-1} \cdot \pi) = a$. First, observe that

$$p(\beta_{(s_i,\pi_i)}(a_i)) = \pi_i \to \pi = p(\beta_{(s,\pi)}(a))$$

by assumption. Next, observe that

$$\beta_{(s,\pi)}(a) = \beta_{(s,\pi)}(b(s^{-1} \cdot \pi)) = \alpha_s(a)(\pi)$$

by definition. Furthermore, $a_i - b(s_i^{-1} \cdot \pi_i) \to 0$ so that by Proposition 3.2 we have $\|a_i - b(s_i^{-1} \cdot \pi_i)\| \to 0$. However, since each $\beta_{(s_i,\pi_i)}$ is an isomorphism, it follows that

$$\|\beta_{(s_i,\pi_i)}(a_i) - \alpha_{s_i}(b)(\pi_i)\| = \|\beta_{(s_i,\pi_i)}(a_i - b(s_i^{-1} \cdot \pi_i))\| = \|a_i - b(s_i^{-1} \cdot \pi_i)\| \to 0.$$

Because α is continuous with respect to the topology of pointwise convergence we have $\alpha_{s_i}(b) \to \alpha_s(b)$ so that

$$\|\alpha_{s_i}(b)(\pi_i) - \alpha_s(b)(\pi_i)\| \leq \|\alpha_{s_i}(b) - \alpha_s(b)\| \to 0.$$

Combining the previous two calculations we can conclude that given $\epsilon > 0$ eventually

$$\|\beta_{(s_i,\pi_i)}(a_i) - \alpha_s(b)(\pi_i)\| < \epsilon.$$

Clearly $\alpha_s(b)(\pi_i) \to \alpha_s(b)(\pi)$ so that it follows from Proposition 3.2 that $\beta_{(s_i,\pi_i)}(a_i) \to \beta_{(s,\pi)}(a)$.

Therefore we have an action β of G on A and as such we can form the crossed product $A \rtimes_\beta G$ as the completion of $\Gamma_c(G, r^*\mathcal{A})$. We claim that $A \rtimes_\beta G$ is isomorphic to $A \rtimes_\alpha H$. Given $f \in \Gamma_c(G, r^*\mathcal{A})$ define

$$\Phi(f)(s)(\pi) = f(s, \pi)$$

for all $s \in H$ and $\pi \in \widehat{A}$. Fix $s \in H$. It is clear that $\Phi(f)(s)(\pi) \in A(\pi)$ for all $\pi \in \widehat{A}$ and that $\Phi(f)(s)$ is continuous and compactly supported. Therefore, after identifying A with $\Gamma_0(\widehat{A}, \mathcal{A})$, we get an element $\Phi(f)(s) \in A$. Suppose $s_i \to s$ and, to the contrary, that $\Phi(f)(s_i) \not\to \Phi(f)(s)$ in $\Gamma_0(\widehat{A}, \mathcal{A})$. This implies that there exists $\epsilon > 0$ and, after passing to a subnet and relabeling, $\pi_i \in \widehat{A}$ such that

$$\|f(s_i, \pi_i) - f(s, \pi_i)\| \geq \epsilon \tag{4.7}$$

for all i. Let K be the projection of $\operatorname{supp} f$ onto $\widehat{A}$. If (4.7) is to hold we must have

$\pi_i \in K$ for all i. However, K is compact so that we may pass to another subnet, relabel, and find $\pi \in K$ so that $\pi_i \to \pi$. However, we now have, using the continuity of f,

$$f(s_i, \pi_i) - f(s, \pi_i) \to f(s, \pi) - f(s, \pi) = 0.$$

It follows from Proposition 3.2 that this contradicts (4.7) so that we must have had $\Phi(f)(s_i) \to \Phi(f)(s)$. Since the support of $\Phi(f)$ is contained in the projection of $\operatorname{supp} f$ to H we have $\Phi(f) \in C_c(H, A)$. Next, suppose $f_i \to f$ with respect to the inductive limit topology. Let K be the compact set which eventually contains the support of the f_i. Then the projection of K to H eventually contains $\operatorname{supp} \Phi(f_i)$. Furthermore, we have, for all $g \in \Gamma_c(G, r^*\mathcal{A})$,

$$\|\Phi(g)\|_\infty = \sup_{s \in H} \|\Phi(g)(s)\| = \sup_{s \in H} \sup_{\pi \in \widehat{A}} \|\Phi(g)(s)(\pi)\| = \sup_{(s,\pi) \in G} \|g(s, \pi)\| = \|g\|_\infty. \quad (4.8)$$

It follows that $\Phi(f_i) \to \Phi(f)$ with respect to the inductive limit topology.

Next, recall that the Haar system on G is given by $\lambda^\pi = \lambda \times \delta_\pi$ where λ is Haar measure and δ_π is the Dirac delta measure at π. We compute for $f, g \in \Gamma_c(G, r^*\mathcal{A})$,

$$\begin{aligned}
\Phi(f * g)(s)(\pi) = f * g(s, \pi) &= \int_G f(t, \rho)\beta_{(t,\rho)}(g((t, \rho)^{-1}(s, \pi)))d(\lambda \times \delta_\pi)(t, \rho) \\
&= \int_H f(t, \pi)\beta_{(t,\pi)}(g(t^{-1}s, t^{-1} \cdot \pi))d\lambda(t) \\
&= \int_H \Phi(f)(t)(\pi)\beta_{(t,\pi)}(\Phi(g)(t^{-1}s)(t^{-1} \cdot \pi))d\lambda(t) \\
&= \int_H (\Phi(f)(t)\alpha_t(\Phi(g)(t^{-1}s)))(\pi)d\lambda(t) \\
&= (\Phi(f) * \Phi(g))(s)(\pi).
\end{aligned}$$

Observe that since "evaluation at π" is given by the quotient map $A \to A(\pi)$ and since this map is bounded and linear we may move it through the integral in the last equality of the above calculation. We also have

$$\begin{aligned}
\Phi(f^*)(s)(\pi) = f^*(s, \pi) &= \beta_{(s,\pi)}(f((s^{-1}, s^{-1} \cdot \pi)^*)) \\
&= \beta_{(s,\pi)}(\Phi(f)(s^{-1})(s^{-1} \cdot \pi))^* = (\alpha_s(\Phi(f)(s^{-1}))(\pi))^* \\
&= \alpha_s(\Phi(f)(s^{-1})^*)(\pi) = \Phi(f)^*(s)(\pi).
\end{aligned}$$

It follows that Φ is a $*$-homomorphism which is continuous in the inductive limit topology. Therefore Proposition 3.133 implies that Φ is bounded with respect to the universal norm and extends to a $*$-homomorphism from $A \rtimes_\beta G$ into $A \rtimes_\alpha H$. We need to show that $\operatorname{ran} \Phi$ is dense in $C_c(H, A)$. Let $D = \Gamma_c(\widehat{A}, \mathcal{A})$ be viewed as a

subset of A and observe that D is dense. Consider the set of sums of elementary tensors $C_c(H) \odot D \subset C_c(H, A)$. By viewing $C_c(G, A)$ as a set of sections of a trivial bundle we can use Proposition 3.11 to show that $C_c(H) \odot D$ is dense in $C_c(H, A)$. First, $C_c(H) \odot D$ is clearly closed under the action of $C_0(H)$. Furthermore, given $s \in H$ choose $\phi \in C_c(H)$ such that $\phi(s) = 1$. Then $\phi \otimes d(s) = d$ for all $d \in D$ and it clear that $C_c(H) \odot D$ is "fibrewise dense." It follows that $C_c(H) \odot D$ is dense in $C_c(H, A)$ with respect to the uniform norm. However, if $\sum_j \phi_i^j \otimes d_i^j \to f$ then we can choose ψ which is one on $\operatorname{supp} f$ and zero off a neighborhood of $\operatorname{supp} f$. We then have $\sum_j \psi\phi_i^j \otimes d_i^j \to f$ with respect to the inductive limit topology. Alternatively we could just cite [Wil07, Lemma 1.87] and skip the previous argument. Now, given $\phi \otimes a \in C_c(H) \odot D$ we define $f(s, \pi) = \phi(s)a(\pi)$ and we have chosen ϕ and a so that $f \in \Gamma_c(G, r^*\mathcal{A})$. Furthermore we clearly have $\Phi(f)(s)(\pi) = \phi(s)a(\pi) = \phi \otimes a(s)(\pi)$. It follows that $C_c(H) \odot D \subset \operatorname{ran}\Phi$ and therefore $\operatorname{ran}\Phi$ is dense in $C_c(H, A)$ with respect to the inductive limit topology, with respect to the I-norm, and with respect to the universal norm.

We would like to show that Φ extends to an isomorphism. First, observe that Φ is clearly injective on $\Gamma_c(G, r^*\mathcal{A})$ so that we can define an inverse map $\Psi = \Phi^{-1} : \operatorname{ran}\Phi \to \Gamma_c(G, r^*\mathcal{A})$ given for $f \in \operatorname{ran}\Phi$ by $\Psi f(s, \pi) = f(s)(\pi)$. It is straightforward to use the fact that Φ is an injective $*$-homomorphism to show that that $\operatorname{ran}\Phi$ is a $*$-subalgebra of $A \rtimes_\alpha H$ and that Ψ is a $*$-homomorphism. Furthermore, given $f \in \operatorname{ran}\Phi$ we calculate

$$\begin{aligned}\int_G \|\Psi(f)(s, \rho)\| d(\lambda \times \delta_\pi)(s, \rho) &= \int_H \|f(s)(\pi)\| d\lambda(s) \\ &\le \int_H \|f(s)\| d\lambda(s) \le \|f\|_I.\end{aligned}$$

It is a similar task to show that

$$\int_G \|\Psi(f)((s, \rho)^{-1}) d(\lambda \times \delta_\pi)(s, \rho) \le \|f\|_I$$

and it follows that Ψ is I-norm decreasing.[1] In particular, we have for all $f \in \operatorname{ran}\Phi$

$$\|\Psi(f)\| \le \|f\|_I.$$

Since $\operatorname{ran}\Phi$ is dense in $C_c(H, A)$ with respect to the I-norm we can extend Ψ from $\operatorname{ran}\Phi$ to all of $C_c(H, A)$. Notice that Ψ now maps into the crossed product $A \rtimes_\beta G$. Furthermore, since the operations are I-norm continuous and Ψ is a $*$-homomorphism

[1] Since we are viewing group crossed products as special cases of groupoid crossed products we use the I-norm on $C_c(G, A)$ and not the L^1-norm.

on ran Φ it follows that Ψ is a I-norm decreasing $*$-homomorphism on all of $C_c(H, A)$. It follows from Proposition 3.133 that Ψ is bounded with respect to the universal norm and we can extend it to a $*$-homomorphism from $A \rtimes_\alpha H$ into $A \rtimes_\beta G$. Furthermore, since Ψ and Φ are inverses on a dense subset, they must be inverses everywhere. It follows that Φ is an isomorphism and $A \rtimes_\alpha H$ and $A \rtimes_\beta G$ are isomorphic.

Remark 4.8. We used some tricky arguments to show that the two crossed products in Example 4.7 are isomorphic. It is possible, and a useful exercise, to show that Φ is an isomorphism by proving that given a covariant representation (π, U) of (A, H, α) then $\pi \rtimes U \circ \Phi$ is equivalent to a covariant representation of (A, G, α) and vice-versa. To construct the covariant representation of (A, G, α) use π as the representation of A. To get a representation of G, view $C^*(G)$ as the crossed product $C_0(\widehat{A}) \rtimes_{\mathrm{lt}} H$. Then extend π to a representation of $C_0(\widehat{A})$, sitting inside the multiplier algebra of A, and form the covariant representation $\overline{\pi} \rtimes U$ of $C_0(\widehat{A}) \rtimes_{\mathrm{lt}} H \cong C^*(G)$. Renault's Decomposition theorem then gives the desired representation of G. In order to go the other direction, basically perform this process in reverse, obtaining the representation of H as part of a covariant decomposition of the representation of $C_0(\widehat{A}) \rtimes_{\mathrm{lt}} H$ coming from the representation of G. Of course, there are a lot of technicalities to work through and this could just be taken as another example of why it's preferable not to work directly with covariant representations.

We also show that Example 4.7 has a nice converse, further strengthening the notion that Example 4.7 provides an alternate method for viewing groupoid crossed products as generalizing the group case.

Proposition 4.9. *Suppose A has Hausdorff spectrum X and that (H, X) is a transformation group. Let $G = H \ltimes X$ be the associated transformation groupoid. If β is an action of G on A then*

$$\alpha_s(a)(\pi) = \beta_{(s,\pi)}(a(s^{-1} \cdot \pi))$$

defines an action of H on A. Furthermore $A \rtimes_\alpha H$ and $A \rtimes_\beta G$ are isomorphic.

Proof. We view A as sections of the associated bundle $\mathcal{A}$. If $a \in A$ then it is clear enough that $\alpha_s(a)$ defines a continuous section of $\mathcal{A}$. Now, if $\epsilon > 0$ then

$$\{\pi : \|\alpha_s(a)(\pi)\| \geq \epsilon\} = \{\pi : \|a(s^{-1} \cdot \pi)\| \geq \epsilon\} = s \cdot \{\pi : \|a(\pi)\| \geq \epsilon\}.$$

Since $\{\pi : \|a(\pi)\| \geq \epsilon\}$ is compact, it follows that $\{\pi : \|\alpha_s(a)(\pi)\| \geq \epsilon\}$ is compact as well. Thus $\alpha_s(a)$ vanishes at infinity and $\alpha_s(a) \in A$. Furthermore since each β is a $*$-homomorphism it is straightforward to show that α_s is a $*$-homomorphism. Next observe that if e is the unit of H then

$$\alpha_e(a)(\pi) = \beta_{(e,\pi)}(a(\pi)) = a(\pi).$$

Thus $\alpha_e = \mathrm{id}$. Furthermore for $s, t \in H$ we have

$$\begin{aligned}\alpha_{st}(a)(\pi) = \beta_{(st,\pi)}(a(t^{-1}s^{-1}\cdot\pi)) &= \beta_{(s,\pi)}(\beta_{(t,s^{-1}\cdot\pi)}(a(t^{-1}s^{-1}\cdot\pi)))\\ &= \beta_{(s,\pi)}(\alpha_t(a)(s^{-1}\cdot\pi)) = \alpha_s(\alpha_t(a))(\pi).\end{aligned}$$

This shows that α_s is an automorphism of A and that $\alpha : H \to \mathrm{Aut}(A)$ is a homomorphism. Finally, fix $a \in A$, let $s_i \to s$ in H and suppose $\alpha_{s_i}(a)$ does not converge to $\alpha_s(a)$. Then, by passing to a subsequence, there exists $\epsilon > 0$ and $\pi_i \in X$ such that

$$\|\alpha_{s_i}(a)(\pi_i) - \alpha_s(a)(\pi_i)\| = \|\beta_{(s_i,\pi_i)}(a(s_i^{-1}\cdot\pi_i)) - \beta_{(s,\pi_i)}(a(s^{-1}\cdot\pi_i))\| \geq \epsilon \tag{4.9}$$

for all i. Consider the compact set $K = \{\rho : \|a(\rho)\| \geq \epsilon/2\}$. If both $s_i^{-1}\cdot\pi_i$ and $s^{-1}\cdot\pi_i$ are not in K then

$$\|\alpha_{s_i}(a)(\pi_i) - \alpha_s(a)(\pi_i)\| \leq \|a(s_i^{-1}\cdot\pi_i)\| + \|a(s^{-1}\cdot\pi_i)\| < \epsilon$$

which contradicts (4.9). There are two cases to consider. If $s^{-1}\cdot\pi_i \in K$ infinitely often then we can pass to a subsequence, multiply by s, and find $\pi \in X$ such that $\pi_i \to \pi$. In the other case $s_i^{-1}\cdot\pi_i$ is in K infinitely often. Therefore we can pass to a subsequence, multiply by the sequence $s_i \to s$, and find $\pi \in X$ such that $\pi_i \to \pi$. In either case we have

$$\begin{aligned}\beta_{(s,\pi_i)}(a(s^{-1}\cdot\pi_i)) &\to \beta_{(s,\pi)}(a(s^{-1}\cdot\pi)), \quad \text{and}\\ \beta_{(s_i,\pi_i)}(a(s_i^{-1}\cdot\pi_i)) &\to \beta_{(s,\pi)}(a(s^{-1}\cdot\pi)).\end{aligned}$$

It follows that

$$\|\alpha_{s_i}(a)(\pi_i) - \alpha_s(a)(\pi_i)\| \to 0$$

which contradicts (4.9). Thus α is a strongly continuous map and is an action of H on A.

Next, given $\pi \in X$ consider its factorization π' to $A(\pi)$. Then

$$s\cdot\pi(a) = (s, s\cdot\pi)\cdot\pi(a) = \pi'(\beta_{(s^{-1},\pi)}(a(s\cdot\pi))) = \pi'(\alpha_s^{-1}(a)(\pi)) = \pi(\alpha_s^{-1}(a))$$

In other words, the action of G on X induced by α is exactly the original action of H on X so that G is the transformation groupoid associated to both actions. Furthermore, as in Example 4.7, there is an action β' of G on A induced by α. However it is easy to see that $\beta' = \beta$. Thus it follows from Example 4.7 that $A \rtimes_\beta G$ is isomorphic to $A \rtimes_\alpha H$. □

Remark 4.10. As was mentioned before, Example 4.7 and Proposition 4.9 are an effective way of viewing groupoid crossed products as generalizing group crossed products.

However, since we require the algebra to have Hausdorff spectrum, this is only a partial generalization. In order to make this work for general C^*-algebras we would need to be able to work with the transformation groupoid associated to the action of the group on the primitive ideal space. This would require us to expand our notion of groupoid dynamical system to include non-Hausdorff groupoids with non-Hausdorff unit spaces that act on bundles over non-Hausdorff spaces, and that seems difficult.

4.2 Groupoid Algebras

In this section we explore another important special case of the groupoid crossed product. We start with the following observation.

Proposition 4.11. *Suppose X is a locally compact Hausdorff space. Then $C_0(X)$ is a $C_0(X)$-algebra with $C_0(X)(x) \cong \mathbb{C}$ for all $x \in X$. Furthermore, the bundle associated to $C_0(X)$ is (isomorphic to) $X \times \mathbb{C}$.*

Proof. It is straightforward to show that $C_0(X)$ is a $C_0(X)$-algebra with the action given by left multiplication. Let I_x be the ideal such that $C_0(X)(x) = C_0(X)/I_x$ and let J_x be the ideal of all functions which vanish at x. It is clear that $I_x \subset J_x$. Let e_l be an approximate unit for $C_0(X)$. Then given $f \in J_x$ we have, by definition, $f \cdot e_l \in I_x$ for all l. It follows that $f \in I_x$. Thus $I_x = J_x$ and it is now straightforward to show that the map $f \mapsto f(x)$ factors to an isomorphism of $C_0(X)(x)$ with $\mathbb{C}$. Let $\mathcal{C}$ denote the bundle associated to $C_0(X)$. We would like to show that $\mathcal{C} \cong X \times \mathbb{C}$. It follows from Corollary 3.31 that it suffices to show that there is a $C_0(X)$-linear isomorphism from $C_0(X)$ onto $\Gamma_0(X, X \times \mathbb{C})$. However it is clear that $\Gamma_0(X, X \times \mathbb{C})$ can be identified with $C_0(X)$ so that the desired isomorphism follows. □

Proposition 4.12. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system. Then there is an action of G on $C_0(G^{(0)})$ given by the collection of functions $\mathrm{id}_\gamma : C_0(G^{(0)})(s(\gamma)) \to C_0(G^{(0)})(r(\gamma))$ such that*

$$\mathrm{id}_\gamma(z) = z$$

for all $z \in \mathbb{C}$ and $\gamma \in G$.

Proof. First observe that the bundle associated to $C_0(G^{(0)})$ as a $C_0(G^{(0)})$-algebra is $G^{(0)} \times \mathbb{C}$. Let us be a bit more formal and define $\mathrm{id}_\gamma : C_0(G^{(0)})(s(\gamma)) \to C_0(G^{(0)})(r(\gamma))$ by $\mathrm{id}_\gamma(s(\gamma), z) = (r(\gamma), z)$. Observe that id_γ is an isomorphism. Furthermore, it is clear that $\mathrm{id}_{\gamma\eta} = \mathrm{id}_\gamma \circ \mathrm{id}_\eta$ if $s(\gamma) = r(\eta)$. The only thing left to check is the continuity condition and this is obvious. □

Thus, given any groupoid we have a natural dynamical system associated to that groupoid. We make the following

Definition 4.13. Let G be a second countable locally compact Hausdorff groupoid G with Haar system λ. We define the *groupoid C^*-algebra* to be the crossed product $C_0(G) \rtimes_{\mathrm{id}} G$ and use the notation $C^*(G)$.

Remark 4.14. While the notation doesn't reflect this, the groupoid algebra $C^*(G)$ depends on the choice of Haar system. When it matters the notation $C^*(G, \lambda)$ is often used. Whether or not the groupoid C^*-algebra is, up to isomorphism, independent of the choice of the Haar system is an open question. On the other hand, it is an immediate corollary to Theorem 4.26 that up to Morita equivalence $C^*(G)$ is independent of the choice of Haar system.

Of course, since the groupoid C^*-algebra is just a crossed product we immediately recover the following from Proposition 3.129.

Corollary 4.15. *Suppose G is a second countable, locally compact Hausdorff groupoid with a Haar system. Then the universal norm is a norm and is dominated by the I-norm. Furthermore, the completion $C^*(G)$ is a separable C^*-algebra.*

The following remark shows how the groupoid C^*-algebra is an extension of the group C^*-algebra.

Remark 4.16. Let G be a second countable locally compact Hausdorff group. Then the the "groupoid" C^*-algebra associated to G is just $C^*(G) = \mathbb{C} \rtimes_{\mathrm{id}} G$. However, it follows from Section 4.1 that this is just a group crossed product. We then cite [Wil07, Example 2.33] to see that this is exactly the group C^*-algebra associated to G. Because the group C^*-algebra is obtained in this way as a group crossed product with $\mathbb{C}$, we often refer to group C^*-algebras as the "scalar" case of group crossed products. The same terminology is used for groupoids. This is further justified by the fact that $C_0(G^{(0)})$ is the simplest $C_0(G^{(0)})$-algebra possible and that the fibres are all isomorphic to $\mathbb{C}$.

We make the following straightforward observations.

Proposition 4.17. *Given a separable locally compact Hausdorff groupoid G with Haar system λ, the section algebra $\Gamma_c(G, G^{(0)} \times \mathbb{C})$ can be identified with $C_c(G)$. Furthermore the operations on $C_c(G)$ become*

$$f * g(\gamma) = \int_G f(\eta) g(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta) \quad \textit{and} \quad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

and the I-norm reduces to

$$\|f\|_I = \max\left\{ \sup_{u \in G^{(0)}} \int_G |f(\gamma)| d\lambda^u(\gamma), \sup_{u \in G^{(0)}} \int_G |f(\gamma)| d\lambda_u(\gamma) \right\}.$$

Proof. In light of the identification made in Remark 3.36 it is clear that any section $f \in \Gamma_c(G, r^*(G^{(0)} \times \mathbb{C}))$ is of the form $f(\gamma) = (r(\gamma), \tilde{f}(\gamma))$ where $\tilde{f} \in C_c(G)$. Thus by identifying $\tilde{f}$ with f we can identify $C_c(G)$ with $\Gamma_c(G, r^*(G^{(0)} \times \mathbb{C}))$. The rest of the claims made in the proposition follow immediately once you recall that the action of G on $C_0(G^{(0)})$ is given by the identity map. □

We continue by considering the covariant representations of $(C_0(G^{(0)}), G, \mathrm{id})$.

Proposition 4.18. *Let G be a second countable locally compact Hausdorff groupoid with a Haar system and $(G^{(0)} * \mathfrak{H}, \mu, U)$ a unitary representation of G. Then the representation $L_\mu : C_0(G^{(0)}) \to L^2(G^{(0)} * \mathfrak{H}, \mu)$ such that $L_\mu(f)\phi(u) = f(u)\phi(u)$ for all $f \in C_0(G^{(0)})$ and $\phi \in \mathcal{L}^2(G^{(0)} * \mathfrak{H}, \mu)$ is $C_0(G^{(0)})$-linear and $(G^{(0)} * \mathfrak{H}, \mu, L_\mu, U)$ is a covariant representation of $(C_0(G^{(0)}), G, \mathrm{id})$. Furthermore, every covariant representation of $(C_0(G^{(0)}), G, \mathrm{id})$ is of this form.*

What's more, given a unitary representation U of G as above, the integrated form of $(G^{(0)} * \mathfrak{H}, \mu, L_\mu, U)$, also denoted by U, is given by

$$U(f)h(\gamma) = \int_G f(\gamma)U_\gamma h(s(\gamma))\Delta(\gamma)^{-\frac{1}{2}} d\lambda^u(\gamma) \tag{4.10}$$

for $f \in C_c(G)$ and $h \in \mathcal{L}^2(G^{(0)} * \mathfrak{H}, \mu)$. Finally, (4.10) *is characterized by*

$$(U(f)h, k) = \int_G (f(\gamma)U_\gamma h(s(\gamma)), k(r(\gamma)))\Delta(\gamma)^{-\frac{1}{2}} d\nu(\gamma) \tag{4.11}$$

where ν is the measure induced by μ.

Proof. Since $L_\mu(f)$ is just a multiplication operator it is straightforward to show that L_μ is a representation of $C_0(G^{(0)})$. Let $\{e_l\}$ be a special orthogonal fundamental sequence for $X * \mathfrak{H}$ and view e_l as an element of $L^2(X * \mathfrak{H}, \mu)$. Given $\phi \in L^2(G^{(0)} * \mathfrak{H}, \mu)$ if $(L_\mu(f)e_l, \phi) = 0$ for all l and $f \in C_c(G^{(0)})^+$ then

$$\int_{G^{(0)}} f(u)(e_l(u), \phi(u))d\mu(u) = 0$$

for all l and $f \in C_c(G^{(0)})^+$. However, it follows that $(e_l(u), \phi(u)) \neq 0$ almost everywhere. We can choose a single null set N such that this is true for all l given $u \notin N$. Since $\{e_l(u)\}$ is a basis for $\mathcal{H}(u)$ it follows that $\phi(u) = 0$ almost everywhere. This is enough to show that L_μ is nondegenerate. Finally, since $L_\mu(f)$ is exactly the diagonal operator associated to f, it is clear that L_μ is $C_0(G^{(0)})$-linear. Consider its associated decomposition

$$L_\mu = \int_{G^{(0)}}^{\oplus} \pi_u d\mu(u).$$

Then, at least for μ-almost every μ, we know that π_u is a nondegenerate representation of $C_0(G^{(0)})(u) = \mathbb{C}$. However, any nondegenerate representation of $\mathbb{C}$ must map 1 onto the identity operator $\mathbf{1}$. Thus, we have $\pi_u(u, z) = z\mathbf{1}$ almost everywhere. We may as well change the decomposition of L_μ on a null set and assume this is always true. However, we now have, given $z \in \mathbb{C}$ and $\gamma \in G$,

$$U_\gamma \pi_{s(\gamma)}(s(\gamma), z) = zU_\gamma = \pi_{r(\gamma)}(\mathrm{id}_\gamma(s(\gamma), z))U_\gamma.$$

Thus (L_μ, U) is a covariant representation of $(C_0(G^{(0)}), G, \mathrm{id})$.

Now suppose $(\mu, G^{(0)} * \mathfrak{H}, \pi, U)$ is a covariant representation of $(C_0(G^{(0)}), G, \mathrm{id})$. Then, by definition, $(\mu, G^{(0)} * \mathfrak{H}, U)$ is a unitary representation of G. All we have to do is show that $\pi = L_\mu$. Find a decomposition $\pi = \int_{G^{(0)}}^{\oplus} \pi_u d\mu(u)$ and observe that π_u is nondegenerate almost everywhere. Then as before we must have $\pi_u(u, z) = z\mathbf{1}$ for μ-almost all u. It follows that, given $f \in C_0(X)$ and $\phi \in \mathcal{L}^2(G^{(0)} * \mathfrak{H}, \mu)$,

$$\pi(f)\phi(u) = \pi_u(f(u))\phi(u) = f(u)\phi(u) = L_\mu(f)\phi(u)$$

μ-almost everywhere. Hence $\pi(f) = L_\mu(f)$ and we are done.

For the last part of the theorem suppose U is a unitary representation and L_μ is the associated representation of $C_0(G^{(0)})$ with $L_\mu = \int_{G^{(0)}}^{\oplus} \pi_u d\mu(u)$ as above so that $\pi_u(z) = z\mathbf{1}$ for all $z \in \mathbb{C}$ and $u \in G^{(0)}$. Then the integrated representation $L_\mu \rtimes U$, denoted by U in the statement of the proposition, is given by

$$U(f)\phi(u) = \int_G \pi_u(f(\gamma))U_\gamma h(s(\gamma))\Delta(\gamma)^{-\frac{1}{2}}\lambda^u(\gamma) = \int_G f(\gamma)U_\gamma h(s(\gamma))\Delta(\gamma)^{-\frac{1}{2}}\lambda^u(\gamma).$$

The final part of the proposition follows from the fact that that $L_\mu \rtimes U$ is characterized by (3.23). $\square$

The last thing we will do is restate Theorem 3.119 for groupoids. This "special case" is actually Renault's Proposition 4.2 and is used to prove the more general Theorem 3.119 [MW08, Section 7].

Theorem 4.19 (Renault's Disintegration Theorem)**.** *Suppose that $\mathcal{H}_0$ is a dense subspace of a complex Hilbert space $\mathcal{H}$ and that u is a homomorphism from $C_c(G)$ into the algebra of linear maps on $\mathcal{H}_0$ such that*

(a) $\{u(f)h : f \in C_c(G), h \in \mathcal{H}_0\}$ *is dense in* $\mathcal{H}$,

(b) *for each* $h, k \in \mathcal{H}_0$,

$$f \mapsto (u(f)h, k)$$

is continuous in the inductive limit topology, and

(c) *for each $f \in C_c(G)$ and all $h, k \in \mathcal{H}_0$*

$$(u(f)h, k) = (h, u(f^*)k).$$

Then $u(f)$ is bounded and extends to a bounded operator $U(f)$ on $\mathcal{H}$ such that U is an I-norm decreasing $*$-representation of $C_c(G)$. Furthermore, there is a unitary representation $(\mu, G^{(0)} * \mathfrak{H}, V)$ such that U is equivalent to the integrated form of V.

Proof. This follows immediately from Theorem 3.119 once you identify unitary representations of G with covariant representations of $(C_0(G^{(0)}), G, \alpha)$ via Proposition 4.18. □

From here we obtain the following corollary, which is really just a restatement of Proposition 3.131.

Corollary 4.20. *Suppose G is a second countable locally compact Hausdorff groupoid with a Haar system. Then the universal norm on $C_c(G)$ is also given by*

$$\|f\| = \sup\left\{ \|U(f)\| : \begin{array}{l} U \text{ is an } I\text{-norm decreasing} \\ *\text{-representation of } C_c(G) \end{array} \right\},$$
$$\|f\| = \sup\left\{ \|U(f)\| : \begin{array}{l} U \text{ is a } *\text{-representation of } C_c(G) \text{ which is} \\ \text{continuous in the inductive limit topology} \end{array} \right\}.$$

It follows that any (possibly degenerate) $*$-representation of $C_c(G)$ which is either I-norm decreasing or continuous with respect to the inductive limit topology is bounded with respect to the universal norm and extends to a representation of $C^*(G)$.

Now it's time for a simple example.

Example 4.21. Suppose X is a second countable locally compact Hausdorff space and we view X as a "cotrivial" groupoid as in Example 1.14. Then X has a Haar system given by the Dirac delta measures δ_x as in Example 1.26. Thus we can form the groupoid algebra $C^*(X)$. However, since integration against δ_x is evaluation at x it is easy to see that $\|f\|_I = \|f\|_\infty$. Thus the I-norm is a C^*-norm and the completion of $C_c(X)$ with respect to the I-norm is clearly $C_0(X)$. By choosing a faithful representation π of $C_0(X)$ it follows from Corollary 4.20 that

$$\begin{aligned} \|f\| &= \sup\{\|\pi(f)\| : \pi \text{ is } I\text{-norm decreasing}\} \\ &\geq \|\pi(f)\| = \|f\|_I = \|f\|_\infty. \end{aligned}$$

It follows that in this case the universal norm is the uniform norm and that the groupoid algebra $C^*(X)$ is just $C_0(X)$.

The point of all of this is twofold. First, groupoid C^*-algebras are much easier to work with than groupoid crossed products because the associated functions are scalar valued and it will be useful to have the simplified operations and representation theory written down. The second is the following remark.

Remark 4.22. Groupoid C^*-algebras have been around for much longer than groupoid crossed products have. It is important to show that viewing groupoid C^*-algebras as special cases of crossed products gets us back to the older definitions. In [Ren80, Chapter II] and [Muh, Chapter 3] the groupoid C^*-algebra is defined to be a universal completion of $C_c(G)$. In the first reference we complete with respect the norm

$$\|f\| = \sup\{\|U(f)\| : U \text{ is an } I\text{-norm decreasing representation}\} \tag{4.12}$$

and in the second we complete with respect to the norm

$$\|f\| = \sup\{\|U(f)\| : U \text{ is a unitary representation}\}. \tag{4.13}$$

However, it is clear once one sorts through the references that the convolution algebra $C_c(G)$ is the same in all three cases. In light of Proposition 4.18, (4.13) is exactly the universal norm and Corollary 4.20 shows it is equal to (4.12). Thus, in both cases the completion of $C_c(G)$ gives us the same algebra as Definition 4.13.

Moving on, we would like to make a connection between groupoid isomorphisms and groupoid algebras. The following proposition is useful and also entirely unsurprising.

Proposition 4.23. *Suppose G and H are second countable locally compact Hausdorff groupoids with Haar systems λ and β respectively. If $\phi : G \to H$ is a groupoid isomorphism such that $\phi_*\lambda^u = \beta^{\phi(u)}$, that is, such that*

$$\int_G f(\phi(\gamma))d\lambda^u(\gamma) = \int_H f(\gamma)d\beta^{\phi(u)}(\gamma)$$

for all $f \in C_c(H)$, then the map $\Phi : C_c(H) \to C_c(G)$ defined by $\Phi(f) = f \circ \phi$ extends to an isomorphism of $C^*(H)$ onto $C^*(G)$.

Remark 4.24. The fact that ϕ has to intertwine the Haar systems is an annoyance. If G has a Haar system and H is isomorphic to G then H has a Haar system induced by the isomorphism. However, it has not been shown that the groupoid C^*-algebra is independent of the choice of Haar measure so that we cannot assume without loss of generality that any two Haar systems on G and H are intertwined. It's also annoying that this proposition does not immediately extend to homomorphisms. The problem is that if ϕ is simply a continuous groupoid homomorphism then there is no reason that $f \circ \phi$ should be compactly supported for a given $f \in C_c(G)$.

Proof. First, $f \circ \phi$ is clearly a continuous compactly supported function so that $\Phi(f)$ is well defined. It is easy to see that this map is linear and we can check that

$$\begin{aligned}\Phi(f * g)(\gamma) = f * g(\phi(\gamma)) &= \int_G f(\eta)g(\eta^{-1}\phi(\gamma))d\beta^{r(\phi(\gamma))}(\eta) \\ &= \int_G f(\phi(\eta))g(\phi(\eta)^{-1}\phi(\gamma))d\lambda^{r(\gamma)}(\eta) \\ &= \Phi(f) * \Phi(g)(\gamma)\end{aligned}$$

and

$$\Phi(f^*)(\gamma) = f^*(\phi(\gamma)) = \overline{f(\phi(\gamma)^{-1})} = \overline{\Phi(f)(\gamma^{-1})} = \Phi(f)^*(\gamma).$$

Thus Φ is a $*$-homomorphism. Since ϕ is a homeomorphism it is easy to show that Φ is continuous in the inductive limit topology. It follows from Corollary 3.134 that Φ extends to a homomorphism from $C^*(H)$ onto $C^*(G)$. However, applying the same argument to ϕ^{-1} gives us a homomorphism from $C^*(G)$ onto $C^*(H)$ which is clearly an inverse for Φ. Thus $C^*(G)$ and $C^*(H)$ are isomorphic. □

We end this section by citing a useful and interesting theorem which makes use of the fact that, as stated in Remark 4.22, viewing groupoid algebras as crossed products brings you back to the more classical object.

Remark 4.25. We assume that the reader is familiar with Morita equivalence, imprimitivity bimodules, pre-Hilbert A-spaces and the like. A good reference for this material is [RW98, Chapters 1,2,3].

Theorem 4.26 (Groupoid Equivalence Theorem [MRW87, Theorem 2.8])**.** *Suppose G and H are second countable locally compact Hausdorff groupoids with Haar systems λ and β. Then for any (G,H)-equivalence X, $\mathcal{Z}_0 = C_c(X)$ is a pre-$C^*(G) - C^*(H)$-imprimitivity bimodule with operations defined for $f \in C_c(G)$, $g \in C_c(H)$, and $\phi, \psi \in C_c(X)$ by*

$$f \cdot \phi(x) = \int_G f(\gamma)\phi(\gamma^{-1} \cdot x)d\lambda^{r(x)}(\gamma) \tag{4.14}$$

$$\phi \cdot g(x) = \int_H \phi(x \cdot \eta)g(\eta^{-1})d\beta^{s(x)}(\eta) \tag{4.15}$$

$$\langle \phi, \psi \rangle_{C^*(H)}(\eta) = \int_G \overline{\phi(\gamma^{-1} \cdot x)}\psi(\gamma^{-1} \cdot x \cdot \eta)d\lambda^{r(x)}(\gamma) \tag{4.16}$$

$${}_{C^*(G)}\langle \phi, \psi \rangle(\gamma) = \int_H \phi(\gamma^{-1} \cdot x \cdot \eta)\overline{\psi(x \cdot \eta)}d\beta^{s(x)}(\eta) \tag{4.17}$$

where x in (4.15) is any element of X such that $s(x) = r(\eta)$ and x in (4.17) is any element of X such that $r(x) = r(\gamma)$. In particular, the completion $\mathcal{Z}$ of $\mathcal{Z}_0$ is a $C^*(G) - C^*(H)$-imprimitivity bimodule and $C^*(G)$ and $C^*(H)$ are Morita equivalent.

Remark. This isn't so much a proof as it is an explanation of how this statement of the Groupoid Equivalence Theorem can be obtained from the statement in the reference. Those readers not interested in chasing down citations can skip ahead. Except for the explicit description of the bimodule operations, Theorem 4.26 is as stated in [MRW87, Theorem 2.8]. The operations themselves can be deduced by scanning through the two pages following the statement of the theorem itself. □

This equivalence theorem is quite nice and is extended to groupoid crossed products in Section 5.1.

4.3 Group Bundles

In this section we present another special case of groupoid crossed products which will play an important role in Sections 5.4 and 6.3. In particular, we will be interested in what happens when a groupoid group bundle S acts on a C^*-algebra. Suppose S is a group bundle over $S^{(0)}$, A is a $C_0(S^{(0)})$-algebra and $\mathcal{A}$ is the upper-semicontinuous bundle associated to A. Now suppose α is an action of S on A and consider the restriction of α to S_u for $u \in S^{(0)}$. Given $t \in S_u$, since $r(t) = s(t) = u$, we have $\alpha_t : A(u) \to A(u)$. It follows from the fact that α preserves the groupoid operation that $\alpha|_{S_u} : S_u \to \mathrm{Aut}(A(u))$ is an automorphism. Furthermore the continuity condition on α implies that $\alpha|_{S_u}$ is continuous onto the topology of pointwise convergence. Putting this all together we have the following

Proposition 4.27. *Suppose (A, S, α) is a separable groupoid dynamical system and S is a group bundle. Then $(A(u), S_u, \alpha|_{S_u})$ is a group dynamical system for each $u \in G^{(0)}$.*

Proof. This follows from the above discussion and Proposition 4.2. □

At this point we have a bundle of groups acting on a bundle of C^*-algebras and it is not surprising that the resulting crossed product is a bundle of group crossed products. However, this proposition is not as easy as it looks and takes some effort to prove.

Proposition 4.28. *Suppose (A, S, α) is a separable groupoid dynamical system and S is a group bundle. Then $A \rtimes_\alpha S$ is a $C_0(S^{(0)})$-algebra with the action defined for $\phi \in C_0(S^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{A})$ by*

$$\Phi(\phi)f(s) = \phi \cdot f(s) := \phi(p(s))f(s).$$

Furthermore the restriction map from $\Gamma_c(S, p^*\mathcal{A})$ to $C_c(S_u, A(u))$ factors to an isomorphism of $A \rtimes_\alpha S(u)$ onto $A(u) \rtimes_{\alpha|_{S_u}} S_u$.

Remark 4.29. It follows that if S is a group bundle acting on A then there is an upper-semicontinuous bundle associated to the crossed product $A \rtimes S$ such that, up to isomorphism, the fibres are the group crossed products $A(u) \rtimes S_u$. For this reason we will sometimes refer to $A \rtimes S$ as a crossed product bundle, a bundle crossed product, or occasionally, a crossed bundle product.

Before we prove Proposition 4.28 we need a series of lemmas.

Lemma 4.30 ([Wil07, Lemma 8.3])**.** *Suppose A is a C^*-algebra and $T \in M(A)$ is such that $T(ab) = aT(b)$ for all $a, b \in A$. Then $T \in ZM(A)$.*

Proof. Let $S \in M(A)$ and $a, b \in A$. Then

$$ST(ab) = S(aT(b)) = S(a)T(b) = T(S(a)b) = TS(ab).$$

This suffices as A^2 is dense in A. $\square$

Lemma 4.31. *The uniform limit of a net of upper-semicontinuous functions is upper-semicontinuous.*

Proof. This is straightforward once you pick the "right" definition of upper semicontinuous. Recall that $f : X \to \mathbb{R}$ is upper-semicontinuous if and only if $\{x : f(x) < a\}$ is open for all $a \in \mathbb{R}$. We claim this is equivalent to the condition that given $x_0 \in X$ and $\epsilon > 0$ there exists an open neighborhood U of x_0 such that $x \in U$ implies $f(x) \leq f(x_0) + \epsilon$. Let's start by proving the forward direction. Given $x_0 \in X$ and $\epsilon > 0$ the set $\{x : f(x) < f(x_0) + \epsilon\}$ is open by assumption and obviously gives us the desired neighborhood of x_0. Now let's tackle the reverse direction. Suppose $a \in \mathbb{R}$ and $f(x_0) < a$. Then let $0 < \epsilon < a - f(x_0)$ and find a neighborhood U of x_0 as above. Then if $x \in U$ we have

$$f(x) \leq f(x_0) + \epsilon < a$$

This is enough to show that $\{x : f(x) < a\}$ is open.

Now suppose $f_i \to f$ uniformly and f_i is upper-semicontinuous for all i. Then given $\epsilon > 0$ and $x_0 \in X$ choose I such that $\|f_I - f\|_\infty < \epsilon/3$ and U such that $x \in U$ implies that $f_I(x) \leq f_I(x_0) + \epsilon/3$. We then have

$$f(x) = f(x) - f_I(x) + f_I(x) \leq \frac{2\epsilon}{3} + f_I(x_0) = \frac{2\epsilon}{3} + f(x_0) + f_I(x_0) - f(x_0) \leq f(x_0) + \epsilon. \quad \square$$

The following lemma is interesting in its own right. Notice that it holds for any groupoid, not just group bundles.

Lemma 4.32. *Suppose (A, G, α) is a separable dynamical system and G has Haar system λ. Then the function*

$$u \mapsto \int_G \|f(\gamma)\| d\lambda^u(\gamma)$$

is upper-semicontinuous for all $f \in \Gamma_c(G, r^*\mathcal{A})$.

Proof. Given $f \in \Gamma_c(G, r^*\mathcal{A})$ define $\lambda(f) : G^{(0)} \to \mathbb{R}$ by

$$\lambda(f)(u) = \int_G \|f(\gamma)\| d\lambda^u(\gamma).$$

If $\phi \otimes a$ is an elementary tensor in $C_c(G) \odot A$ then

$$\lambda(\phi \otimes a)(u) = \int_G |\phi(\gamma)| d\lambda^u(\gamma) \|a(u)\|.$$

It follows from the continuity of the Haar system and the fact that $u \mapsto \|a(u)\|$ is upper-semicontinuous that $\lambda(\phi \otimes a)$ is upper semicontinuous. Now suppose $f \in \Gamma_c(G, r^*\mathcal{A})$. Then there exists a set of elementary tensors $\{\phi_i^j \otimes a_i^j\}$ such that $k_i = \sum_j \phi_i^j \otimes a_i^j$ converges to f with respect to the inductive limit topology and therefore with respect to the I-norm. Now, $\lambda(k_i) = \sum_j \lambda(\phi_i^j \otimes a_i^j)$, and it is straightforward to show that sums of upper-semicontinuous functions are upper-semicontinuous. Hence, $\lambda(k_i)$ is upper-semicontinuous. It now follows quickly from the computation

$$\begin{aligned}
\left| \int_G \|f(\gamma)\| d\lambda^u(\gamma) - \int_G \|g(\gamma)\| d\lambda^u(\gamma) \right| &\le \int_G \big| \|f(\gamma)\| - \|g(\gamma)\| \big| d\lambda^u(\gamma) \\
&\le \int_G \|f(\gamma) - g(\gamma)\| d\lambda^u(\gamma) \\
&\le \|f - g\|_I
\end{aligned}$$

that $\lambda(k_i) \to \lambda(f)$ uniformly and the result follows from Lemma 4.31. $\square$

Proof of Proposition 4.28. Given $\phi \in C_0(S^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{A})$ define $\Phi(\phi)f$ as in the statement of the proposition. It is easy to see that $\Phi(\phi)f \in \Gamma_c(S, p^*\mathcal{A})$ and that $\Phi(\phi)$ is linear as a function on $\Gamma_c(S, p^*\mathcal{A})$. Next observe that given $\phi, \psi \in C_0(S^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{A})$ we have

$$\Phi(\phi)\Phi(\psi)f(s) = \phi(p(s))\Phi(\psi)f(s) = \phi(p(s))\psi(p(s))f(s) = \Phi(\phi\psi)f(s). \tag{4.18}$$

Thus Φ preserves the multiplication on $C_0(S^{(0)})$. We need to extend $\Phi(\phi)$ to an element of the multiplier algebra. The following slick proof is thanks to Dana Williams.

Recall that multipliers on $A \rtimes S$ are adjointable operators on $A \rtimes S$ where we view $A \rtimes S$ as a right $A \rtimes S$-module with the operations

$$f \cdot g = f * g, \qquad\qquad \langle f, g\rangle = f^* * g.$$

We start by showing that, for all $f, g \in \Gamma_c(S, p^*\mathcal{A})$,

$$\Phi(\phi)(f * g)(s) = \phi(p(s)) \int_G f(t)\alpha_t(g(t^{-1}s))d\beta^{p(s)}(t) = (\Phi(\phi)f) * g(s).$$

Thus $\Phi(\phi)$ is $A \rtimes S$-linear on $\Gamma_c(S, p^*\mathcal{A})$. Next we show that

$$\langle \Phi(\phi)f, g\rangle = \langle f, \Phi(\overline{\phi})g\rangle. \tag{4.19}$$

We compute for $s \in S$ and $u = p(s)$

$$\begin{aligned}\langle \Phi(\phi)f, g\rangle(s) = (\phi \cdot f)^* * g(s) &= \int_S \alpha_t(\phi \cdot f(t^{-1})^*)\alpha_t(g(t^{-1}s))d\beta^u(t)\\ &= \int_S \alpha_t(\overline{\phi(u)}f(t^{-1})^* g(t^{-1}s))d\beta^u(t) = \int_S \alpha_t(f(t^{-1})^*\overline{\phi} \cdot g(t^{-1}s))d\beta^u(t)\\ &= f^* * \overline{\phi} \cdot g(s) = \langle f, \Phi(\overline{\phi})g\rangle.\end{aligned}$$

Now extend Φ to the unitization $C_0(S^{(0)})^1$ by setting $\Phi(\phi + \lambda 1)f = \Phi(\phi)f + \lambda f$. It is an easy computation to show that (4.18) and (4.19) extend to all of $C_0(S^{(0)})^1$. Now suppose $\phi \in C_0(G^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{A})$. It follows from the C^*-identity that $\|\langle f, f\rangle\| = \|f\|^2$. If we want to show that $\|\Phi(\phi)f\| \le \|\phi\|_\infty \|f\|$ then, because the norm respects the ordering in a C^*-algebra, it suffices to show that as C^*-algebra elements $\langle \phi \cdot f, \phi \cdot f\rangle \le \|\phi\|_\infty^2 \langle f, f\rangle$. However, using (4.19) we find that this is equivalent to showing

$$\begin{aligned}0 &\le \|\phi\|_\infty^2 \langle f, f\rangle - \langle \Phi(\phi)f, \Phi(\phi)f\rangle\\ &= \langle \|\phi\|_\infty^2 f, f\rangle + \langle \Phi(\overline{\phi})\Phi(\phi)f, f\rangle\\ &= \langle \Phi(\|\phi\|_\infty^2 1 - \overline{\phi}\phi)f, f\rangle.\end{aligned}$$

Since general C^*-algebra theory assures us that $\|\phi\|_\infty^2 1 - \overline{\phi}\phi$ is positive in $C_0(S^{(0)})^1$ it follows that there is some $\xi \in C_0(S^{(0)})^1$ such that $\xi^*\xi = \|\phi\|_\infty^2 1 - \overline{\phi}\phi$. We now compute

$$\begin{aligned}\langle \Phi(\|\phi\|_\infty^2 1 - \overline{\phi}\phi)f, f\rangle &= \langle \Phi(\xi^*)\Phi(\xi)f, f\rangle\\ &= \langle \Phi(\xi)f, \Phi(\xi)f\rangle\\ &= (\Phi(\xi)f)^* * (\Phi(\xi)f) \ge 0\end{aligned}$$

It follows that $\Phi(\phi)$ is a bounded linear operator on $\Gamma_c(S, p^*\mathcal{A})$ with norm less than $\|\phi\|_\infty$ and as such extends to an operator on $A \rtimes_\alpha S$. Since $\Phi(\phi)$ is $A \rtimes S$-linear on a dense subset, it is $A \rtimes S$-linear everywhere. Furthermore (4.19) implies that $\Phi(\phi)$ is adjointable with $\Phi(\phi)^* = \Phi(\overline{\phi})$. Hence $\Phi(\phi) \in M(A \rtimes_\alpha S)$. It also follows from (4.18) and (4.19) that Φ is a $*$-homomorphism. We would like to show that Φ maps into the center. Using Lemma 4.30, it suffices to show that $\Phi(\phi)(f * g) = f * \Phi(\phi)g$ for all $f, g \in \Gamma_c(S, p^*\mathcal{A})$. We compute for $s \in S$ and $u = p(s)$

$$\begin{aligned}\Phi(\phi)(f * g) = \phi \cdot (f * g)(s) &= \int_S \phi(u) f(t) \alpha_t(g(t^{-1}s)) d\beta^u(t) \\ &= \int_S f(t)\alpha_t(\phi \cdot g(t^{-1}s)) d\beta^u(t) = f * \Phi(\phi)g(s).\end{aligned}$$

The last thing we need to do to show that $A \rtimes S$ is a $C_0(S^{(0)})$-algebra is show that the set $\Phi(C_0(S^{(0)})) \cdot A \rtimes S$ is dense in $A \rtimes S$. However, given $f \in \Gamma_c(S, p^*\mathcal{A})$ we can find $\phi \in C_c(S^{(0)})$ such that ϕ is one on $p(\operatorname{supp} f)$ and in this case $\Phi(\phi)f = f$. It follows immediately that Φ is nondegenerate.

Now we tackle the second part of the assertion. Fix $u \in S^{(0)}$ and define $R : \Gamma_c(S, p^*\mathcal{A}) \to C_c(S_u, A(u))$ by restriction. It is clear that R is a well-defined linear map and it is trivial to show that R is a $*$-homomorphism. Furthermore R is uniform norm decreasing and it is straightforward to show that it is continuous with respect to the inductive limit topology. Thus Corollary 3.134 implies that R extends to a map $R : A \rtimes_\alpha S \mapsto A(u) \rtimes_{\alpha|_{S_u}} S_u$. Now, given $\phi \in C_c(S_u)$ and $a \in A(u)$ find $b \in A$ such that $b(u) = a$. Next, since S is second countable, we can use the Tietze Extension Theorem to extend ϕ so that $\phi \in C_c(S)$. However it is clear that $R(\phi \otimes b) = \phi \otimes a$ and therefore $\operatorname{ran} R$ contains the elementary tensors. Treating $C_c(S_u, A(u))$ as sections of the trivial bundle it follows from Corollary 3.45 that $\operatorname{ran} R$ is dense in $C_c(S_u, A(u))$. Since the range is dense, R must be onto.

Let

$$I_u = \overline{\operatorname{span}}\{\phi \cdot f : \phi \in C_0(S^{(0)}), \phi(x) = 0, f \in \Gamma_c(S, p^*\mathcal{A})\}$$

and recall that, by definition, $A \rtimes S(u) = A \rtimes S/I_u$. We would like to show that $I_u = \ker R$. This next part of the proof is inspired by [RW88, Lemma 2.3]. If $\phi(u) = 0$ then for all $s \in S_u$ we have $\phi \cdot f(s) = \phi(u)f(s) = 0$. It follows that $R(\phi \cdot f) = 0$ and that $I_u \subset \ker R$. Next, suppose that $f \in \Gamma_c(S, p^*\mathcal{A})$ and $R(f) = 0$. Since $s \mapsto \|f(s)\|$ is upper-semicontinuous, given $\epsilon > 0$ we can find an open neighborhood U of S_u such that $\|f(s)\| < \epsilon$ for all $s \in U$. Choose $\phi \in C_c(S^{(0)})$ such that $0 \leq \phi \leq 1$, $\phi(u) = 0$, and ϕ is one on $p(\operatorname{supp} f \setminus U)$. Then $\|f - \phi \cdot f\|_\infty < \epsilon$ and $\operatorname{supp}(\phi \cdot f) \subset \operatorname{supp} f$. This is enough to show, after a straightforward argument, that $f \in I_u$. Now, let π be a representation of $A \rtimes S$ such that $\ker \pi = I_u$. It follows from the above that if $f, g \in \Gamma_c(S, p^*\mathcal{A})$ such that $R(f) - R(g) = 0$ then we have $f - g \in \ker \pi$. Hence, we

can define a representation ρ of $C_c(S_u, A(u))$ by $\rho(R(f)) = \pi(f)$. It is easy to show, using the fact that π and R preserve the operations, that ρ does as well. Furthermore, given $f \in \Gamma_c(S, p^*\mathcal{A})$ for any $\phi \in C_0(S^{(0)})^+$ such that $\phi(u) = 1$ we have

$$\|\rho(R(f))\| = \|\pi(f)\| = \|\pi(\phi \cdot f)\| \leq \|\phi \cdot f\| \leq \|\phi \cdot f\|_I \tag{4.20}$$

Let $B = \{\phi \in C_0(S^{(0)})^+ : \phi(u) = 1\}$, $M = \inf_{\phi \in B} \|\phi \cdot f\|_I$ and observe that (4.20) implies $\|\rho(R(f))\| \leq M$. We make the following claim.

Claim. Given f and M as above we have

$$M = \|R(f)\|_I = \max\left\{\int_S \|f(s)\| d\beta^u(s), \int_S \|f(s^{-1})\| d\beta^u(s)\right\}. \tag{4.21}$$

Proof of Claim. First observe that, almost by definition, $\|R(f)\|_I \leq M$. Suppose $\epsilon > 0$. It follows from Lemma 4.32 that the function

$$v \mapsto \int_S \|f(s)\| d\beta^v(s)$$

is upper-semicontinuous. As such there exists a relatively compact neighborhood U of u such that $v \in U$ implies

$$\int_S \|f(s)\| d\beta^v(s) \leq \int_S \|f(s)\| d\beta^u(s) + \epsilon \leq \|R(f)\|_I + \epsilon.$$

Since $s \mapsto f(s^{-1})$ defines an element of $\Gamma_c(S, p^*\mathcal{A})$ we can use Lemma 4.32 again to see that the function

$$v \mapsto \int_S \|f(s^{-1})\| d\beta^v(s)$$

is upper-semicontinuous. As such there exists a relatively compact neighborhood V of u such that $v \in V$ implies

$$\int_S \|f(s^{-1})\| d\beta^v(s) \leq \int_S \|f(s^{-1})\| d\beta^u(s) + \epsilon \leq \|R(f)\|_I + \epsilon.$$

Choose $\phi \in C_c(S^{(0)})$ such that $0 \leq \phi \leq 1$, $\phi(u) = 1$ and $\phi(v) = 0$ for all $v \notin U \cap V$. Now if $v \in U \cap V$ we have

$$\phi(v) \int_S \|f(s)\| d\beta^v(s) \leq \phi(v)(\|R(f)\|_I + \epsilon) \leq \|R(f)\|_I + \epsilon, \quad \text{and}$$
$$\phi(v) \int_S \|f(s^{-1})\| d\beta^v(s) \leq \phi(v)(\|R(f)\|_I + \epsilon) \leq \|R(f)\|_I + \epsilon.$$

If $v \notin U \cap V$ we have

$$\phi(v) \int_S \|f(s)\| d\beta^v(s) = 0 \leq \|R(f)\|_I + \epsilon, \quad \text{and}$$
$$\phi(v) \int_S \|f(s^{-1})\| d\beta^v(s) = 0 \leq \|R(f)\|_I + \epsilon.$$

In any case, we certainly have

$$\begin{aligned} \|\phi \cdot f\|_I &= \max\left\{ \sup_{v \in S^{(0)}} \phi(v) \int_S \|f(s)\| d\beta^v(s), \sup_{v \in \in S^{(0)}} \phi(v) \int_S \|f(s^{-1})\| d\beta^v(s) \right\} \\ &\leq \|R(f)\|_I + \epsilon. \end{aligned}$$

Since we were able to find such a ϕ for any ϵ it follows that $M \leq \|R(f)\|_I$. The claim follows. □

At this point we have shown that $\|\rho(R(f))\| \leq \|R(f)\|_I$ for any $f \in \Gamma_c(S, p^*\mathcal{A})$ and as such ρ is bounded on $C_c(S_u, A(u))$ with respect to the I-norm. We have already asserted that ρ is a $*$-homomorphism. Thus, we can use Proposition 3.131 to extend ρ to the entire group crossed product $A(u) \rtimes S_u$. Furthermore since $\rho \circ R = \pi$ on a dense subset, this identity extends to all of $A \rtimes S$. It follows that $\ker R \subset \ker \pi = I_u$. This shows that $\ker R = I_u$ and hence restriction factors to an isomorphism of $A \rtimes S(u)$ onto $A(u) \rtimes S_u$. □

Remark 4.33. One important consequence of Proposition 4.28 is that the irreducible representations of $A \rtimes S$ are well behaved. To elaborate, Proposition 3.22 and the second part of Proposition 4.28 says that, as a set, the spectrum $(A \rtimes S)^\wedge$ can be identified with the disjoint union $\coprod_{u \in S_u} (A(u) \rtimes S_u)^\wedge$. In other words, every irreducible representation of the crossed product bundle $A \rtimes S$ is lifted from an irreducible covariant representation of the group crossed product $A(u) \rtimes S_u$ for some $u \in S^{(0)}$. This will be an important theme in Section 6.3.

We end this section with a restriction of Proposition 4.28 to the scalar case. This is the proposition used in Section 2.2 to give the total space of the dual bundle a topology.

Proposition 4.34. *If S is a locally compact Hausdorff group bundle with a Haar system then $C^*(S)$ is a $C_0(S^{(0)})$-algebra. Furthermore the restriction map from $C_c(S)$ onto $C_c(S_u)$ factors to an isomorphism from $C^*(S)(u)$ onto the group C^*-algebra $C^*(S_u)$ for all $u \in S^{(0)}$.*

Proof. The groupoid algebra $C^*(S)$ is defined to be the crossed product $C_0(S^{(0)}) \rtimes_{\mathrm{id}} S$. By Proposition 4.28 the crossed product is a $C_0(S^{(0)})$-algebra and the restriction map

factors to an isomorphism from $C^*(S)(u)$ onto the crossed product $C_0(S^{(0)})(u) \rtimes_{\mathrm{id}} S_u$. However, $C_0(S^{(0)})(u) = \mathbb{C}$ and $\mathrm{id}|_{S_u}$ is still the identity action so that $\mathbb{C} \rtimes S_u$ is equal to $C^*(S_u)$ by definition. The fact that $C^*(S_u)$ is also the group algebra associated to S_u follows from Remark 4.16. $\square$

4.4 Transformation Groupoid Algebras

Our last special case of groupoid crossed products comes from the notion of a groupoid action introduced in Section 1.2. This section is particularly important because it also introduces a very natural groupoid action associated to any crossed product. We start with the following definition.

Definition 4.35. Suppose a second countable locally compact Hausdorff groupoid G with a Haar system acts continuously on a second countable locally compact Hausdorff space X. The *transformation groupoid C^*-algebra* of (G, X) is defined to be the groupoid C^*-algebra of the transformation groupoid $G \ltimes X$. Furthermore, we use the notation $C^*(G, X) := C^*(G \ltimes X)$. When G acts on the right of X the transformation groupoid C^*-algebra is defined in an analogous way and we use the notation $C^*(X, G) := C^*(X \rtimes G)$.

This definition makes sense because the transformation groupoid has a Haar system whenever G does, by Proposition 1.72.

Remark 4.36. Definition 4.35 is slightly specious in that the notation $C^*(G, X)$ isn't any simpler than $C^*(G \ltimes X)$. However, it does connect back to the notation used for transformation group C^*-algebras. Furthermore, although we won't address it here, $C^*(G, X)$ simplifies to the transformation group C^*-algebra when G is a group, and of course is also the same as the C^*-algebra of the transformation group groupoid in this case.

The ultimate goal of this section will be to show that if G acts on X then G acts on $C_0(X)$ and that the groupoid crossed product in this case is the same as the transformation groupoid C^*-algebra. However, before we can prove this result we need the following lemma. Basically, we need a tool to deal with the topology on the upper-semicontinuous bundles we will encounter later on.

Lemma 4.37. *Suppose X, Y and Z are locally compact Hausdorff spaces and that $\sigma : Y \to X$ and $\tau : Z \to X$ are continuous surjections. Let $Z * Y = \{(z, y) \in Z \times Y : \tau(z) = \sigma(y)\}$. Then the map $\iota : C_0(Z * Y) \to \tau^*(C_0(Y))$ such that $\iota(f)(z)(y) = f(z, y)$ is an isometric isomorphism. Furthermore $\iota(f)$ is compactly supported if f is and ι preserves convergence with respect to the inductive limit topology.*

Proof. Let $\mathcal{C}$ be the upper-semicontinuous bundle associated to $C_0(Y)$ as a $C_0(X)$-algebra and recall that $C_0(Y)(x) = C_0(\sigma^{-1}(x))$. Define $\iota : C_c(Z * Y) \to \Gamma_c(Z, \tau^*\mathcal{C})$ by $\iota(f)(z)(y) = f(z, y)$. It is clear that $\iota(f)(z)$ is a continuous compactly supported function on $\sigma^{-1}(\tau(z))$. Therefore $\iota(f)(z) \in C_0(Y)(\tau(z))$ and ι is a well defined section of $\tau^*\mathcal{C}$. It is also clear that $\iota(f)$ has compact support. Notice that this also verifies the claim that $\iota(f)$ is compactly supported if f is. We need to see that $\iota(f)$ is continuous. We start by demonstrating this in a simpler case. Suppose $g \in C_c(Z)$, $h \in C_c(Y)$ and define $g \otimes h(z, y) = g(z)h(y)$ for all $(z, y) \in Z * Y$. Let $z_i \to z$. We would like to show that $\iota(g \otimes h)(z_i) \to \iota(g \otimes h)(z)$ in $\mathcal{C}$. Since $h \in C_c(X)$ we can view h as a continuous section of $\mathcal{C}$ with $h(x) = h|_{\sigma^{-1}(x)}$ for all $x \in X$ and it follows that $h(\tau(z_i)) \to h(\tau(z))$ in $\mathcal{C}$ since τ is continuous. Furthermore, scalar multiplication is continuous and $g(z_i) \to g(z)$ so we also have

$$g(z_i)h(\tau(z_i)) \to g(z)h(\tau(z))$$

in $\mathcal{C}$. However, it is clear with some thought that $\iota(g \otimes h)(w) = g(w)h(\tau(w)) = g(w)h|_{\sigma^{-1}(\tau(w))}$ for any $w \in Z$ and therefore $\iota(g \otimes h) \in \Gamma_c(Z, \tau^*\mathcal{C})$.

Now suppose we have $f \in C_c(Z * Y)$. Since $Z * Y$ is closed in $Z \times Y$ we can use Lemma 1.71 to extend f to the product space. Next find $g_i^j \in C_c(Z)$ and $h_i^j \in C_c(Y)$ such that $k_i = \sum_j g_i^j \otimes h_i^j \to f$ uniformly. Now let $z_i \to z$ and observe that $\tau(z_i) \to \tau(z)$. We will show that $\iota(f)(z_i) \to \iota(f)(z)$ using Proposition 3.2. Let $\epsilon > 0$ and choose I such that $\|k_I - f\|_\infty < \epsilon$. Since sums of continuous functions are continuous, $\iota(k_I)(z_i) \to \iota(k_I)(z)$ by the previous paragraph. Furthermore given any $w \in Z$ we have

$$\|\iota(k_I)(w) - \iota(f)(w)\|_\infty = \sup_{y \in \sigma^{-1}(\tau(w))} |k_I(w, y) - f(w, y)| \leq \|k_I - f\|_\infty < \epsilon.$$

Since this is true for all z_i and z as well it follows from the last part of Proposition 3.2 that $\iota(f)(z_i) \to \iota(f)(z)$. Thus $\iota(f) \in \Gamma_c(Z, r^*\mathcal{C})$.

Suppose $f \in C_c(Z * Y)$. We calculate

$$\begin{aligned}
\|\iota(f)\|_\infty &= \sup_{z \in Z} \|\iota(f)(z)\|_\infty \\
&= \sup_{z \in Z} \sup_{y \in \sigma^{-1}(\tau(z))} |f(z, y)| = \|f\|_\infty.
\end{aligned}$$

Hence ι is isometric. Next we show that $\operatorname{ran} \iota$ is dense in $\tau^*C_0(X) = \Gamma_0(Z, \tau^*\mathcal{C})$. If $f \in C_c(Z * Y)$ and $g \in C_0(Z)$ then observe that if we define $h(z, y) = g(z)f(z, y)$ then $h \in C_c(Z * Y)$ and

$$(g \cdot \iota(f)(z))(y) = g(z)f(z, y) = \iota(h)(z)(y).$$

It follows that $\operatorname{ran}\iota$ is closed under the action of $C_0(Z)$. Next, fix $z \in G$ and $f \in C_c(\sigma^{-1}(\tau(z)))$. Find $g \in C_c(Y)$ such that $g|_{\sigma^{-1}(\tau(z))} = f$ and $h \in C_c(Z)$ such that $h(z) = 1$. Then clearly $\iota(h \otimes g)(z) = f$ so that $\operatorname{ran}\iota$ is dense fibrewise. It follows from Proposition 3.11 that $\operatorname{ran}\iota$ is dense in $\Gamma_0(Z, \tau^*\mathcal{C})$. Since $\iota : C_c(Z*Y) \to \tau^*C_0(Y)$ is an isometry mapping onto a dense set we can extend ι to an isomorphism $\iota : C_0(Z*Y) \to \tau^*C_0(Y)$. Furthermore, if $f \in C_0(Z*Y)$ then we can find a sequence in $C_c(Z*Y)$ such that $f_i \to f$ uniformly. However, it follows that $\iota(f_i) \to \iota(f)$ uniformly. Thus $\iota(f_i)(z) \to \iota(f)(z)$. This convergence takes place in $C_0(Y)(\tau(z)) = C_0(\sigma^{-1}(\tau(z)))$ and is therefore convergence with respect to the uniform norm. Again this implies that for $y \in \sigma^{-1}(\tau(z))$ we have

$$\iota(f_i)(z)(y) = f_i(z, y) \to \iota(f)(z)(y).$$

However we also have $f_i(z, y) \to f(z, y)$. Thus ι has the desired form on all of $C_0(Z*Y)$.

Finally, observe that if $f_i \to f$ with respect to the inductive limit topology then $\iota(f_i) \to \iota(f)$ uniformly. Furthermore if the supports of f_i are eventually contained in K then the supports of $\iota(f_i)$ are eventually contained in the projection of K onto Z. Thus $\iota(f_i) \to \iota(f)$ with respect to the inductive limit topology. □

Proposition 4.38. *Suppose a second countable, locally compact Hausdorff groupoid G acts on a second countable, locally compact Hausdorff space X. Then $C_0(X)$ is a $C_0(G^{(0)})$-algebra and there is an action of G on $C_0(X)$ given by the maps $\mathrm{lt}_\gamma : C_0(r_X^{-1}(s(\gamma))) \to C_0(r_X^{-1}(r(\gamma)))$ where*

$$\mathrm{lt}_\gamma(f)(y) = f(\gamma^{-1} \cdot y) \tag{4.22}$$

for all $f \in C_0(r_X^{-1}(s(\gamma)))$ and $y \in r_X^{-1}(r(\gamma))$. Furthermore, the groupoid crossed product $C_0(X) \rtimes_{\mathrm{lt}} G$ is isomorphic to $C^*(G \ltimes X)$ and the isomorphism is given on $C_c(G \ltimes X)$ by $\iota(f)(\gamma)(x) = f(\gamma, x)$.

Proof. The fact that $C_0(X)$ is a $C_0(G^{(0)})$-algebra is really just Example 3.17 using the map $r_X : X \to G^{(0)}$. The action is given by $\Phi(f)(g)(x) = f(r(x))g(x)$ and it is straightforward to use the Stone-Weierstrass theorem to show that this makes $C_0(X)$ into a $C_0(G^{(0)})$-algebra. Let I_u be the ideal such that $C_0(X)(u)$ is defined to be $C_0(X)/I_u$ and $I_{r^{-1}(u)}$ the ideal of functions which are zero on $r_X^{-1}(u)$. It is clear from the definition of I_u that $I_u \subset I_{r^{-1}(u)}$. However, given $f \in I_{r^{-1}(u)}$ and $\epsilon > 0$ since $|f|$ is continuous the set $O = \{x \in X : |f| < \epsilon\}$ is open. Choose $g \in C_c(G^{(0)})$ so that $g(u) = 0$ and g is one on $\operatorname{supp} f \setminus U$. It's easy to check that $\|\Phi(g)f - f\|_\infty < \epsilon$. It follows quickly that $f \in I_u$ and $I_{r^{-1}(u)} = I_u$. However, $I_{r^{-1}(u)}$ is the ideal of functions which are zero on some closed set and it follows from classical theory that $C_0(X)/I_{r^{-1}(u)} \cong C_0(r^{-1}(u))$. Thus, we can view the upper-semicontinuous bundle

$\mathcal{C}$ associated to $C_0(X)$ as having fibres $C_0(r_X^{-1}(u))$. We deal with the fact that the topology on $\mathcal{C}$ is mysterious by using Proposition 3.50.

Use Lemma 4.37 to observe that there are two isomorphisms

$$\iota_s : C_0(G * X) \to s^*C_0(X), \qquad\qquad \iota_r : C_0(G \ltimes X) \to r^*C_0(X)$$

where

$$\begin{aligned} G * X &= \{(\gamma, x) \in G \times X : s(\gamma) = r(x)\}, \quad \text{and} \\ G \rtimes X &= \{(\gamma, x) \in G \times X : r(\gamma) = r(x)\}. \end{aligned}$$

We use these isomorphisms to, perhaps foolishly, identify $C_0(G * X)$ with $s^*C_0(X)$ and $C_0(G \ltimes X)$ with $r^*C_0(X)$. We define a map $\mathrm{lt} : s^*C_0(X) \to r^*C_0(X)$ by

$$\mathrm{lt}(f)(\gamma, x) = f(\gamma, \gamma^{-1} \cdot x).$$

It is clear that lt is a $*$-homomorphism and it is easy to construct an inverse for lt using "right translation." Furthermore, if $\phi \in C_0(G)$ we have

$$\phi \cdot \mathrm{lt}(f)(\gamma, x) = \phi(\gamma) f(\gamma, \gamma^{-1} \cdot x) = \mathrm{lt}(\phi \cdot f)(\gamma, x).$$

Thus lt is a $C_0(G)$-linear $*$-isomorphism. As such it defines a family of isomorphisms $\mathrm{lt}_\gamma : C_0(X)(s(\gamma)) \to C_0(X)(r(\gamma))$. Given $f \in C_0(X)(s(\gamma)) = C_0(r_X^{-1}(s(\gamma)))$ extend f to a function $g \in C_0(X)$ and pick $h \in C_c(G)$ such that $h(\gamma) = 1$. By construction $h \otimes g(\gamma) = f$ so that we have, by the definition of lt_γ,

$$\mathrm{lt}_\gamma(f) = \mathrm{lt}_\gamma(h \otimes g(\gamma)) = \mathrm{lt}(h \otimes g)(\gamma).$$

But now we can compute

$$\mathrm{lt}_\gamma(f)(x) = \mathrm{lt}(h \otimes g)(\gamma, x) = h(\gamma) g(\gamma^{-1} \cdot x) = f(\gamma^{-1} \cdot x).$$

Thus the action lt has the desired form. Furthermore, it is now easy to verify that $\mathrm{lt}_\gamma \circ \mathrm{lt}_\eta = \mathrm{lt}_{\gamma\eta}$. It follows from Proposition 3.50 that $(C_0(X), G, \mathrm{lt})$ is a groupoid dynamical system.

Now consider the restriction of the isomorphism ι_r to $\iota : C_c(G \ltimes X) \to \Gamma_c(G, r^*\mathcal{C})$. We already used the fact that ι preserves the pointwise operations. Now we show that it preserves the convolution and involution as well. We compute

$$\iota(f) * \iota(g)(\gamma)(x) = \int_G \iota(f)(\eta)(x)\, \mathrm{lt}_\eta(\iota(g)(\eta^{-1}\gamma))(x) d\lambda^{r(\gamma)}(\eta)$$

$$
\begin{aligned}
&= \int_G f(\gamma, x) g(\eta^{-1}\gamma, \eta^{-1} \cdot x) d\lambda^{r(\gamma)}(\eta) \\
&= \int_G f(\gamma, x) g((\eta, x)^{-1}(\gamma, x)) d\lambda^{r(x)}(\eta) \\
&= f * g(\gamma, x) = \iota(f * g)(\gamma)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\iota(f)^*(\gamma)(x) &= \mathrm{lt}_\gamma(\iota(f)(\gamma^{-1})^*)(x) = \overline{f(\gamma^{-1}, \gamma^{-1} \cdot x)} \\
&= \overline{f((\gamma, x)^{-1})} = f^*(\gamma, x) \\
&= \iota(f^*)(\gamma)(x).
\end{aligned}
$$

Thus ι is a $*$-homomorphism with respect to the crossed product operations as well. Furthermore we proved in Lemma 4.37 that ι is continuous with respect to the inductive limit topology. It then follows from Corollary 3.134 that ι extends to a homomorphism $I : C^*(G \ltimes X) \to C_0(X) \rtimes_{\mathrm{lt}} G$. We would like to see that this map is an isomorphism. Let $j : \operatorname{ran} \iota \to C_c(G \ltimes X)$ be the inverse of ι so that $j(f)(\gamma, x) = f(\gamma)(x)$. Then j is clearly onto, injective, and a $*$-homomorphism. Given $x \in X$ we have

$$
\begin{aligned}
\int_G |j(f)(\gamma, x)| d\lambda^{r(x)}(\gamma) &= \int_G |f(\gamma)(x)| d\lambda^{r(x)}(\gamma) \\
&\leq \int_G \|f(\gamma)\|_\infty d\lambda^{r(x)}(\gamma) \leq \|f\|_I.
\end{aligned}
$$

Similarly we calculate that

$$
\int_G |j(f)(\gamma^{-1}, \gamma^{-1} \cdot x)| d\lambda^{r(x)}(\gamma) \leq \int_G \|f(\gamma^{-1})\|_\infty d\lambda^{r(x)}(\gamma) \leq \|f\|_I.
$$

Once we recall the definition of the Haar system on $G \ltimes X$ it follows immediately that j is I-norm decreasing. It is straightforward to show, using the fact that i_r is an isomorphism, that $\operatorname{ran} \iota$ is dense in $\Gamma_c(G, r^*\mathcal{C})$ with respect to the inductive limit topology and with respect to the I-norm. Now, extend j to $\Gamma_c(G, r^*\mathcal{C})$. Then Proposition 3.133 implies that j extends to a $*$-homomorphism $J : C_0(X) \rtimes_{\mathrm{lt}} G \to C^*(G \ltimes X)$. However, J and I are inverses on a dense set so they must be inverses everywhere. It follows that I is an isomorphism of the transformation groupoid C^*-algebra onto the groupoid crossed product. $\square$

There is a "converse" to Proposition 4.38 in that we can identify those crossed products which arise from transformation groupoids. However, the following piece of this construction is necessary for what we will do in Chapter 6 and is very interesting

in its own right.

Proposition 4.39. *If (A, G, α) is a groupoid dynamical system then there is a continuous action of G on $\widehat{A}$ given by $\gamma \cdot \pi = \pi \circ \alpha_\gamma^{-1}$.*

Proof. Since A is a $C_0(G^{(0)})$-algebra it follows from Proposition 3.22 that there is a continuous map $r : \widehat{A} \to G^{(0)}$. Furthermore, as in Proposition 3.22, we view $\widehat{A}$ as being fibred over $G^{(0)}$ so that if $\pi \in \widehat{A}$ with $r(\pi) = u$ then $I_u \subset \ker \pi$ and we can factor π to a representation π' of $A(u)$. Oftentimes we will not distinguish between π and π', but it is important to do so now. Given $\gamma \in G$ we know $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$ so that if $r(\pi) = s(\gamma)$ we can define $\gamma \cdot \pi \in \widehat{A}$ by

$$\gamma \cdot \pi(a) = \pi'(\alpha_\gamma^{-1}(a(r(\gamma)))).$$

Of course when we factor $\gamma \cdot \pi$ to $A(r(\gamma))$ we get $(\gamma \cdot \pi)' = \pi' \circ \alpha_\gamma^{-1}$ as desired. Given $u \in G^{(0)}$ we know $\alpha_u = \mathrm{id}$ so it is clear that $r(\pi) \cdot \pi = \pi$. Furthermore if γ and η are composable then

$$\gamma \cdot (\eta \cdot \pi)(a) = \pi'(\alpha_\eta^{-1}(\alpha_\gamma^{-1}(a(r(\gamma))))) = (\gamma\eta) \cdot \pi(a).$$

All that is left is to check that the action is continuous. Suppose $\gamma_i \to \gamma$ and $\pi_i \to \pi$ such that $s(\gamma_i) = r(\pi_i)$ for all i and $s(\gamma) = r(\pi)$. Let $O_J = \{\pi \in \widehat{A} : J \not\subset \ker \pi\}$ be an open set in $\widehat{A}$ containing $\gamma \cdot \pi$. Recall that every open set is of this form [RW98, Corollary A.28]. Suppose, to the contrary, that $\gamma_i \cdot \pi_i$ is not eventually in O_J. By passing to a subnet and relabeling we can assume $\gamma_i \cdot \pi_i \notin O_J$ for all i. Let $a \in J$ and choose $b \in A$ such that $b(s(\gamma)) = \alpha_\gamma^{-1}(a(r(\gamma)))$. Since the action is continuous, $\alpha_{\gamma_i}^{-1}(a(r(\gamma_i))) \to b(s(\gamma))$. Since the norm is upper-semicontinuous, the set $\{a \in \mathcal{A} : \|a\| < \epsilon\}$ is open for all $\epsilon > 0$. Because $\alpha_{\gamma_i}^{-1}(a(r(\gamma_i))) - b(s(\gamma_i)) \to 0$ we eventually have $\|\alpha_{\gamma_i}^{-1}(a(r(\gamma_i))) - b(s(\gamma_i))\| < \epsilon$ for all $\epsilon > 0$. Hence

$$\|\alpha_{\gamma_i}^{-1}(a(r(\gamma_i))) - b(s(\gamma_i))\| \to 0.$$

Next, $\gamma_i \cdot \pi_i \notin O_J$ for all i so we have

$$\gamma_i \cdot \pi_i(a) = \pi'(\alpha_{\gamma_i}^{-1}(a(r(\gamma_i)))) = 0$$

for all i. Hence

$$\begin{aligned} \|\pi_i(b)\| = \|\pi_i'(b(s(\gamma_i)))\| &= \|\pi_i'(b(s(\gamma_i)) - \alpha_{\gamma_i}^{-1}(a(r(\gamma_i))))\| \\ &\leq \|b(s(\gamma_i)) - \alpha_{\gamma_i}^{-1}(a(r(\gamma_i)))\| \to 0. \end{aligned} \tag{4.23}$$

It is shown in [RW98, Lemma A.30] that the map $\pi \mapsto \|\pi(b)\|$ is lower-semicontinuous

on $\widehat{A}$. In other words given $\epsilon \geq 0$ the set

$$\{\rho \in \widehat{A} : \|\rho(b)\| \leq \epsilon\}$$

is closed. Now, given $\epsilon > 0$ equation (4.23) implies that eventually $\pi_i \in \{\rho \in \widehat{A} : \|\rho(b)\| \leq \epsilon\}$. Since this set is closed and $\pi_i \to \pi$ we know that $\|\pi(b)\| \leq \epsilon$. This is true for all $\epsilon > 0$ so that

$$0 = \pi(b) = \pi'(b(s(\gamma))) = \pi'(\alpha_\gamma^{-1}(a(r(\gamma)))) = \gamma \cdot \pi(a).$$

This is a contradiction since $a \in J$ was arbitrary and we assumed that $\gamma \cdot \pi \in O_J$. It follows that the action of G on $\widehat{A}$ is continuous.[2] □

This allows us to prove that every groupoid action on an *abelian* C^*-algebra comes from a groupoid action on a space, which provides the promised "converse" to Proposition 4.38.

Proposition 4.40. *Suppose X is a locally compact Hausdorff space and $(C_0(X), G, \alpha)$ is a separable dynamical system. Then there is an action of G on X such that α is given by left translation. Consequently $C_0(X) \rtimes_\alpha G \cong C^*(G, X)$.*

Proof. We know from Proposition 4.39 that there is a continuous action of G on $C_0(X)^\wedge$. We can identify $C_0(X)^\wedge$ and X via the Gelfand transform which takes $x \in X$ to "evaluation at x", denoted ev_x, and $\pi \in C_0(X)^\wedge$ to the element $\hat{\pi} \in X$ determined by $f(\hat{\pi}) = \pi(f)$ for $f \in C_0(X)$. Thus, using Proposition 4.39, we can view G as acting on X via the formula

$$\gamma \cdot x = (\gamma \cdot \mathrm{ev}_x)^\wedge = (\mathrm{ev}_x \circ \alpha_\gamma^{-1})^\wedge. \tag{4.24}$$

This action of G on X induces an action of G on $C_0(X)$ via left translation as in Proposition 4.38. That is, given $f \in C_0(X)(s(\gamma)) = C_0(r_X^{-1}(s(\gamma)))$ we define for $x \in r_X^{-1}(r(\gamma))$,

$$\mathrm{lt}_\gamma(f)(x) = f(\gamma^{-1} \cdot x).$$

However, we compute that

$$\begin{aligned} \mathrm{lt}_\gamma(f)(x) &= f(\gamma^{-1} \cdot x) = f((\mathrm{ev}_x \circ \alpha_\gamma)^\wedge) \\ &= \mathrm{ev}_x(\alpha_\gamma(f)) = \alpha_\gamma(f)(x) \end{aligned}$$

Thus $\mathrm{lt}_\gamma = \alpha_\gamma$ and α is given by left translation. It follows from Proposition 4.38 that $C^*(G, X) \cong C_0(X) \rtimes_\alpha G$. □

[2] The author finds it amusing that this proof uses both upper and lower semicontinuity.

4.4.1 Stone Von-Neumann Theorem

While we are on the subject of groupoid actions and groupoid spaces we may as well discuss one of the most basic groupoid actions, the action of G on itself by translation. Recall from Example 1.64 that the action is defined by $\gamma \cdot \eta = \gamma\eta$. This is a very well behaved example and we can give a nice description of its associated transformation groupoid C^*-algebra. This theorem is a generalization of the Stone Von-Neumann theorem for group crossed products [Wil07, Theorem 4.24]. It is slightly surprising that it can be extended to groupoids with such generality.

Theorem 4.41. *Suppose G is a second countable locally compact Hausdorff groupoid with a Haar system and let G act on the left of itself by translation. Then there is an isomorphism Φ from $C^*(G,G)$ onto $\mathcal{K}(\mathcal{Z})$ where $\mathcal{Z}$ is the completion of the pre-Hilbert $C_0(G^{(0)})$-module $\mathcal{Z}_0 = C_c(G)$ given the operations*

$$z \cdot \phi(\gamma) = z(\gamma)\phi(s(\gamma)) \qquad\qquad \langle w, z\rangle(u) = \int_G \overline{w(\gamma)} z(\gamma) d\lambda_u(\gamma).$$

Furthermore, given $f \in C_c(G \ltimes G)$ and $z \in Z_0$ we have

$$\Phi(f)z(\gamma) = \int_G f(\eta,\gamma) z(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta). \tag{4.25}$$

Proof. First, consider the fact that if we let $G^{(0)}$ act on the right of G by the trivial action then G is a strong, principal, right $G^{(0)}$-space. As such we can form the imprimitivity groupoid $H = G^{G^{(0)}}$. Since the action is trivial, we have

$$H = G * G = \{(\gamma,\eta) \in G \times G : s(\gamma) = s(\eta)\}$$

and the operations are just $(\gamma,\eta)(\eta,\zeta) = (\gamma,\zeta)$ and $(\gamma,\eta)^{-1} = (\eta,\gamma)$. It follows from Proposition 1.93 that H acts on the left of G and that with this action G is a $(H, G^{(0)})$-equivalence. Furthermore, once we untangle the definitions, it follows that the action of H on G is defined by $(\gamma,\eta) \cdot \eta = \gamma$. Now G has a Haar system by assumption, and since the restriction of the range and source maps to $G^{(0)}$ are obviously open it follows from Proposition 1.95 that H has a Haar system μ defined by

$$\int_H f(\eta,\gamma) d\mu^\zeta(\eta,\gamma) = \int_G f(\zeta,\gamma) d\lambda_{s(\zeta)}(\gamma).$$

Furthermore, we saw in Example 4.21 that $C^*(G^{(0)})$ is just $C_0(G^{(0)})$ and the Haar system is given by the Dirac delta measures $\{\delta_u\}$.

Since both $G^{(0)}$ and H have Haar systems we can use Theorem 4.26 to view $\mathcal{Z}_0 = C_c(G)$ as a pre-$C^*(H) - C_0(G^{(0)})$-imprimitivity bimodule. In particular $\mathcal{Z}_0$ is

a pre-Hilbert $C_0(G^{(0)})$-module and we can compute that the operations are given for $z \in C_c(G)$ and $\phi \in C_0(G^{(0)})$ by

$$\begin{aligned} z \cdot \phi(\gamma) &= \int_{G^{(0)}} z(\gamma \cdot u)\phi(u^{-1}) d\delta_{s(\gamma)}(u) \\ &= z(\gamma)\phi(s(\gamma)), \end{aligned}$$

and by picking $\gamma \in G$ such that $s(\gamma) = u$,

$$\begin{aligned} \langle z, w\rangle_{C_0(G^{(0)})}(u) &= \int_H \overline{z((\eta,\zeta)^{-1} \cdot \gamma)} w((\eta,\zeta)^{-1} \cdot \gamma \cdot u) d\mu^\gamma(\eta,\zeta) \\ &= \int_G \overline{z((\zeta,\gamma) \cdot \gamma)} w((\zeta,\gamma) \cdot \gamma) d\lambda_{s(\gamma)}(\zeta) \\ &= \int_G \overline{z(\zeta)} w(\zeta) d\lambda_u(\zeta). \end{aligned}$$

Thus $\mathcal{Z}_0$ has the appropriate operations and defines the desired Hilbert $C_0(G^{(0)})$-module $\mathcal{Z}$. Furthermore, since $\mathcal{Z}$ is an $C^*(H) - C_0(G^{(0)})$-imprimitivity bimodule it follows from [RW98, Proposition 3.8] that $C^*(H)$ is isomorphic to the compact operators $\mathcal{K}(\mathcal{Z})$ and that the isomorphism is given by $\Psi(f)(z) = f \cdot z$. Thus we can compute for $f \in C_c(H)$ and $z \in C_c(G)$ that

$$\begin{aligned} \Psi(f)(z)(\gamma) = f \cdot z(\gamma) &= \int_H f(\eta,\zeta) z((\eta,\zeta)^{-1} \cdot \gamma) d\mu^\gamma(\eta,\zeta) \\ &= \int_G f(\gamma,\zeta) z(\zeta) d\lambda_{s(\gamma)}(\zeta). \end{aligned}$$

Now, consider the transformation groupoid $G \ltimes G$. Define $\phi : H \to G \ltimes G$ by $\phi(\gamma,\eta) = (\gamma\eta^{-1},\gamma)$. This map is clearly continuous and has a continuous inverse given by $(\gamma,\eta) \mapsto (\eta, \gamma^{-1}\eta)$. Furthermore, we can check that

$$\begin{aligned} \phi((\gamma,\eta)(\eta,\xi)) &= \phi(\gamma,\xi) = (\gamma\xi^{-1},\gamma) \\ &= (\gamma\eta^{-1},\gamma)(\eta\xi^{-1},\eta) = \phi(\gamma,\eta)\phi(\eta,\xi), \quad \text{and} \\ \phi((\gamma,\eta)^{-1}) &= \phi(\eta,\gamma) = (\eta\gamma^{-1},\eta) \\ &= (\gamma\eta^{-1},\gamma)^{-1} = \phi(\gamma,\eta)^{-1}. \end{aligned}$$

Thus ϕ is a groupoid isomorphism. Now recall that the Haar system on $G \ltimes G$ is defined by $\beta^\gamma = \lambda^{r(\gamma)} \times \delta_\gamma$. We can then check that, for $f \in C_c(G \ltimes G)$

$$\int_H f(\phi(\zeta,\eta)) d\mu^\gamma(\zeta,\eta) = \int_G f(\phi(\gamma,\eta)) d\lambda_{s(\gamma)}(\eta) = \int_G f(\gamma\eta^{-1},\gamma) d\lambda_{s(\gamma)}(\eta)$$

$$= \int_G f(\eta,\gamma)d\lambda^{r(\gamma)}(\eta) = \int_{G\ltimes G} f(\eta,\zeta)d\beta^\gamma(\eta,\zeta).$$

Thus ϕ intertwines the Haar systems on $G \ltimes G$ and H. It follows from Proposition 4.23 that the map $\Upsilon : C^*(G) \to C^*(H)$ defined for $f \in C_c(G)$ by $\Upsilon(f) = f \circ \phi$ is an isomorphism. But now we can define an isomorphism $\Phi : C^*(G) \to \mathcal{K}(\mathcal{Z})$ by $\Phi = \Psi \circ \Upsilon$. Furthermore, we can compute for $f \in C_c(G \ltimes G)$ and $z \in C_c(G)$ that

$$\begin{aligned}\Phi(f)z(\gamma) = \Psi(f\circ\phi)z(\gamma) &= \int_G f\circ\phi(\gamma,\zeta)z(\zeta)d\lambda_{s(\gamma)}(\zeta)\\ &= \int_G f(\gamma\zeta^{-1},\gamma)z(\zeta)d\lambda_{s(\gamma)}(\zeta) = \int_G f(\gamma\zeta,\gamma)z(\zeta^{-1})d\lambda^{s(\gamma)}(\zeta)\\ &= \int_G f(\zeta,\gamma)z(\zeta^{-1}\gamma)d\lambda^{r(\gamma)}(\zeta). \qquad \square\end{aligned}$$

Theorem 4.41 is not just a generalization of the Stone-von Neumann theorem in the thematic sense. The classical Stone-von Neumann theorem can be obtained as a corollary.

Corollary 4.42. *Suppose G is a second countable locally compact group. Let M be the representation of $C_0(G)$ on $L^2(G)$ given by multiplication and let L be the left regular representation. Then $M \rtimes L$ is a faithful representation of $C_0(G) \rtimes_{\mathrm{lt}} G$ onto $\mathcal{K}(L^2(G))$.*

Proof. Let us first use Proposition 4.38 to identify $C_0(G) \rtimes_{\mathrm{lt}} G$ with $C^*(G,G) = C^*(G \ltimes G)$. Once we perform this identification the representation $M \rtimes L$ becomes, for $f \in C_c(G \ltimes G)$ and $\phi \in \mathcal{L}^2(G)$,

$$M \rtimes L(f)\phi(s) = \int_G M(f(t))L_t\phi(s)\Delta(t)^{-\frac{1}{2}}dt = \int_G f(t,s)\phi(t^{-1}s)\Delta(t)^{-\frac{1}{2}}dt.$$

Now, since G is a group, $G^{(0)}$ consists of a single point and $G \ltimes G = G \times G$. We can use Theorem 4.41 to construct an isomorphism Φ of $C^*(G \ltimes G)$ with $\mathcal{K}(\mathcal{Z})$ where $\mathcal{Z}$ is the completion of the pre-Hilbert $C_0(G^{(0)})$-module $\mathcal{Z}_0 = C_c(G)$. However $C_0(G^{(0)}) = \mathbb{C}$ so that $\mathcal{Z}$ is a Hilbert space. Furthermore, given $\phi,\psi \in C_c(G)$ we have

$$\langle \phi,\psi\rangle = \int_G \overline{\phi(s)}\psi(s)d\lambda_e(s) = \int_G \overline{\phi(s)}\psi(s)\Delta(s)ds.$$

Now define $U : \mathcal{Z}_0 \to L^2(G)$ by $U\phi(s) = \phi(s)\Delta(s)^{-\frac{1}{2}}$. It is straightforward to show that U extends to a unitary from $\mathcal{Z}$ onto $L^2(G)$ and thus $U\Phi U^*$ is an isomorphism from $C^*(G,G)$ onto $\mathcal{K}(L^2(G))$. Furthermore, given $f \in C_c(G \ltimes G)$ and $\phi \in C_c(G)$ we

have

$$\begin{aligned} U\Phi(f)U^*\phi(s) &= \int_G f(t,s)U^*\phi(t^{-1}s)\Delta^{-\frac{1}{2}}(s)dt \\ &= \int_G f(t,s)\phi(t^{-1}s)\Delta^{-\frac{1}{2}}(t)dt = M \rtimes L(f)\phi(s). \end{aligned}$$

It follows that $M \ltimes L = U^*\Phi(f)U$ so that M is a faithful representation onto $\mathcal{K}(L^2(G))$. $\square$

Chapter 5

Basic Constructions

Half of this chapter consists of technical tools and the other half of capstone results. In particular, Section 5.1 is a mixture of both. We introduce some major results for groupoid crossed products in this section, but we also apply them and show that a transitive groupoid crossed product is Morita equivalent to a group crossed product. Section 5.2 is entirely technical but is essential for Section 6.2. In Section 5.3 we introduce the notion of a unitary action and prove that for unitary actions the crossed product reduces to a tensor product. This result is to be expected. What's more interesting is the definition of locally unitary actions given in Section 5.4. There we show that locally unitary actions give rise to principal bundles and that the exterior equivalence class of the action is characterized by the cohomological invariant of the associated bundle. Furthermore, we show that locally unitary actions can be constructed from any principal bundle.

5.1 Transitive Groupoid Crossed Products

In this section, we will compute the representations of crossed products by transitive groupoids. This work will be essential in Section 6.2 and also generalizes the latter half of [MRW87]. In order to accomplish our goal we will have to introduce the notion of an equivalence bundle for groupoid dynamical systems. These objects are intimidating because they are very complex. However, as we will see, in practice they are fairly easy to work with. Now, we won't do anything other than write down the definition and cite the major theorem for equivalences. The reader is referred to [MW08] for the whole story. It is also mildly helpful to keep in mind that this material is meant to be a generalization of groupoid equivalences, which were introduced in Section 1.2.1.

Remark 5.1. In what follows we use the following notation. Given two bundles

$p : E \to X$ and $q : F \to X$ we define

$$E * F = \{(e, f) \in E \times F : p(e) = q(f)\}.$$

Definition 5.2. Suppose (A, G, α) and (B, H, β) are two separable groupoid dynamical systems with associated upper-semicontinuous bundles $\mathcal{A}$ and $\mathcal{B}$. An *equivalence* between the dynamical systems (A, G, α) and (B, H, β) is an upper-semicontinuous Banach bundle[1] $p_{\mathfrak{H}} : \mathfrak{H} \to X$ over a (G, H)-equivalence X together with $A(r(x)) - B(s(x))$-imprimitivity bimodule structures on each fibre $\mathfrak{H}_x = p_{\mathfrak{H}}^{-1}(x)$ and commuting *strongly* continuous actions of G and H on the left and right, respectively, of $\mathfrak{H}$ such that the following additional properties are satisfied for $e, f \in \mathfrak{H}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, $\gamma \in G$ and $\eta \in H$:

(a) (Continuity) The maps induced by the imprimitivity bimodule inner products from $\mathfrak{H} * \mathfrak{H} \to \mathcal{A}$ and $\mathfrak{H} * \mathfrak{H} \to \mathcal{B}$ are continuous, as are the maps $\mathcal{A} * \mathfrak{H} \to \mathfrak{H}$ and $\mathfrak{H} * \mathcal{B} \to \mathfrak{H}$ induced by the imprimitivity bimodule actions.

(b) (Equivariance) The bundle map $p_{\mathfrak{H}}$ is equivariant with respect to the groupoid actions. In other words, $p_{\mathfrak{H}}(\gamma \cdot e) = \gamma \cdot p_{\mathfrak{H}}(e)$ and $p_{\mathfrak{H}}(e \cdot \eta) = p_{\mathfrak{H}}(e) \cdot \eta$.

(c) (Compatibility) The groupoid actions are compatible with the imprimitivity bimodule structure:

$$\begin{aligned} {}_A\langle \gamma \cdot e, \gamma \cdot f\rangle &= \alpha_\gamma({}_A\langle e, f\rangle) & \qquad \langle e \cdot \eta, f \cdot \eta\rangle_B &= \beta_\eta^{-1}(\langle e, f\rangle_A) \\ \gamma \cdot (a \cdot e) &= \alpha_\gamma(a) \cdot (\gamma \cdot e) & (e \cdot b) \cdot \eta &= (e \cdot \eta) \cdot \beta_\eta^{-1}(b). \end{aligned}$$

(d) (Invariance) The G-action commutes with the B-action on $\mathfrak{H}$ and the H-action commutes with the A-action. That is, $\gamma \cdot (e \cdot b) = (\gamma \cdot e) \cdot b$ and $(a \cdot e) \cdot \gamma = a \cdot (e \cdot \gamma)$.

The only reason anyone is inspired to try and form an equivalence bundle is that, in the same vein as Theorem 4.26, we get the following useful theorem. This result is can be found, in French, in [Ren87, Corollaire 5.4] or, in English, in [MW08, Theorem 5.5].

Theorem 5.3 (Renault's Equivalence Theorem)**.** *Suppose G and H are second countable locally compact Hausdorff groupoids with Haar systems λ_G and λ_H respectively. Furthermore, suppose that (A, G, α) and (B, H, β) are separable dynamical systems and $p_{\mathfrak{H}} : \mathfrak{H} \to X$ is an equivalence between (A, G, α) and (B, H, β). Then $\mathcal{Z}_0 = \Gamma_c(X, \mathfrak{H})$ becomes a pre-$A \rtimes_\alpha G - B \rtimes_\beta H$-imprimitivity bimodule with respect to the*

[1] The symbol $\mathfrak{H}$ is pronounced "humpf" as in "Humpf-Humpf-a-Dumpfer" [Seu55].

following operations for $f \in \Gamma_c(G, r^*\mathcal{A})$, $g \in \Gamma_c(H, r^*\mathcal{B})$ *and* $z, w \in \Gamma_c(X, \mathfrak{E})$*:*

$$f \cdot z(x) := \int_G f(\gamma) \cdot (\gamma \cdot z(\gamma^{-1} \cdot x)) d\lambda_G^{r(x)}(\gamma) \tag{5.1}$$

$$z \cdot g(x) := \int_H (z(x \cdot \eta) \cdot \eta^{-1}) \cdot \beta_\eta(g(\eta^{-1})) d\lambda_H^{s(x)}(\eta) \tag{5.2}$$

$$\langle\langle z, w \rangle\rangle_{B \rtimes_\beta H}(\eta) := \int_G \langle z(\gamma^{-1} \cdot x \cdot \eta^{-1}), w(\gamma^{-1} \cdot x) \cdot \eta^{-1} \rangle_{B(r(\eta))} d\lambda_G^{r(x)}(\gamma) \tag{5.3}$$

$$_{A \rtimes_\alpha G}\langle\langle z, w \rangle\rangle(\gamma) := \int_H {}_{A(r(\gamma))}\langle z(\gamma \cdot x \cdot \eta), \gamma \cdot w(x \cdot \eta) \rangle d\lambda_H^{s(x)}(\eta) \tag{5.4}$$

where x *in* (5.3) *is any element of* X *such that* $s(x) = s(\eta)$ *and* x *in* (5.4) *is any element of* X *such that* $r(x) = s(\gamma)$*. In particular, the completion* $\mathcal{Z}$ *of* $\mathcal{Z}_0$ *is a* $A \rtimes_\alpha G - B \rtimes_\beta H$*-imprimitivity bimodule and* $A \rtimes_\alpha G$ *and* $B \rtimes_\beta H$ *are Morita equivalent.*

This theorem will be essential to our study of induced representations. We can also give a straightforward application in the case of transitive groupoid actions. We will start by citing a deep theorem by Ramsay. The following is a slightly trimmed down transcription of [Ram90, Theorem 2.1].

Theorem 5.4 (Mackey-Glimm Dichotomy)**.** *Let* G *be a second countable locally compact Hausdorff groupoid. Then the following are equivalent:*

(a) *For each* $u \in G^{(0)}$*, the map* $\gamma \cdot S_u \mapsto r(\gamma)$ *of* G_u/S_u *to the orbit* $G \cdot u$ *is a homeomorphism.*

(b) $G^{(0)}/G$ *is a* T_0*-space.*

(c) *Each orbit is locally closed in* $G^{(0)}$*.*

(d) *The quotient topology on* $G^{(0)}/G$ *generates the quotient Borel structure.*

(e) *The quotient Borel structure on* $G^{(0)}/G$ *is countably separated.*

(f) $G^{(0)}/G$ *is almost Hausdorff.*[2]

(g) $G^{(0)}/G$ *is a standard Borel space.*

(h) *The quotient map* $\pi : G^{(0)} \to G^{(0)}/G$ *has a Borel section.*

Remark. This isn't so much a proof as it is an explanation of how this statement of the Mackey-Glimm Dichotomy can be obtained from the statement in the reference.

[2] See Definition 6.18.

Those readers not interested in chasing down citations can skip ahead. The result in [Ram90] is for Polish groupoids. Since second countable locally compact Hausdorff spaces are completely metrizable the result clearly applies. Furthermore, in order to use the full power of [Ram90, Theorem 2.1] we need to know that the orbit groupoid R is a F_σ subset of $G^{(0)} \times G^{(0)}$. However, G is second countable locally compact Hausdorff so that it is the countable union of compact sets, say $\{K_i\}_{i=1}^{\infty}$. Consider the canonical map $\pi : G \to R_P$. Since π is continuous, $\pi(K_i)$ is compact (with respect to the product topology) and therefore closed in $G^{(0)} \times G^{(0)}$. Thus $R = \bigcup_{i=1}^{\infty} \pi(K_i)$ is F_σ. Therefore all fourteen (!) conditions listed in [Ram90] are equivalent and the conditions stated in the theorem are a subset of those. □

Theorem 5.4 is known as the Mackey-Glimm Dichotomy because it says that either a number of nice conditions hold or none of them do. We will primarily be interested in the case where $G^{(0)}/G$ is T_0, but will spend some time on the "other side" of the dichotomy as well. As we will see in Section 7.1, things don't always work as smoothly as they do when Theorem 5.4 is satisfied.

The next proposition makes more sense if you realize that given $\gamma \in G$ conjugation by γ defines an isomorphism from $S_{s(\gamma)}$ onto $S_{r(\gamma)}$. Thus, in a transitive groupoid all of the stabilizer subgroups must be isomorphic.

Proposition 5.5 ([MRW87, Example 2.2]). *Suppose G is a transitive second countable locally compact Hausdorff groupoid. Then given $u \in G^{(0)}$ the space $X = G_u$ is a (G, S_u)-equivalence with respect to the natural actions of G and S_u.*

Proof. Let G act on $X = G_u$ by left translation and S_u act on X by right translation. It is easy to show that X is a free continuous left G-space and a free continuous right S_u-space and that the actions commute. Now suppose $\gamma_i \to \gamma$ in X and $\eta_i\gamma_i \to \zeta$ in X. Then $\eta_i \to \zeta\gamma^{-1}$ so that by Proposition 1.84 the action of G on X is proper. The same argument shows that the action of S_u on X is proper as well. Furthermore since the source map s_X on X maps onto a single point it must be open. We also note that if $\gamma, \eta \in X$ then γ and η^{-1} are composable and that $(\gamma\eta^{-1}) \cdot \eta = \gamma$. Thus the action of G on X is transitive so that s_X factors to a bijection. Next, let $r_X : X \to G^{(0)}$ be the restriction of the range map to X. Given $\gamma, \eta \in X$, if $r(\gamma) = r(\eta)$ then $\gamma^{-1}\eta \in S_u$ so that $\gamma \cdot \gamma^{-1}\eta = \eta$. This shows that r_X factors to a bijection from X/S_u onto $G^{(0)}$. All that is left is to show that r_X is open. This is not necessarily true in the nonseparable case, [MRW87, Example 2.2]. However, we assumed that G is second countable and, since it is transitive, the quotient space $G^{(0)}/G$ is trivial. Clearly $G^{(0)}/G$ is T_0 so that we may use Theorem 5.4 to conclude that the map $\rho : G_u/S_u \to G \cdot u = G^{(0)}$ such that $\rho(\gamma \cdot S_u) = r(\gamma)$ is a homeomorphism. However, r_X is just the composition of ρ with the quotient map from G_u onto G_u/S_u. Since both of these maps are open r_X is open as well. □

Remark 5.6. It is worth noting that you do not need to use the Mackey-Glimm Dichotomy to prove Proposition 5.5. A more elementary proof can be found in [MRW87, Theorem 2.2A,2.2B]. This argument makes use of the Baire Category Theorem and is quite interesting.

We can now use Proposition 5.5 to identify the Morita equivalence class of transitive groupoid crossed products. As before, observe that because G is transitive each of the stabilizer subgroups must be isomorphic. Furthermore, the α_γ guarantee that all of the fibres of $\mathcal{A}$ are isomorphic as well.

Theorem 5.7. *Suppose (A, G, α) is a separable dynamical system and that G is transitive with Haar system λ. Fix $u \in G^{(0)}$ and let β be Haar measure on S_u. Then $\mathcal{X}_0 = C_c(G_u, A(u))$ becomes a pre-$A \rtimes_\alpha G - A(u) \rtimes_{\alpha|_{S_u}} S_u$-imprimitivity bimodule with respect to the following operations for $f \in \Gamma_c(G, r^*\mathcal{A})$, $g \in C_c(S_u, A(u))$ and $z, w \in \mathcal{X}_0$:*

$$f \cdot z(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta))z(\eta^{-1}\gamma)d\lambda^{r(\gamma)}(\eta) \tag{5.5}$$

$$z \cdot g(\gamma) = \int_{S_u} \alpha_s(z(\gamma s)g(s^{-1}))d\beta(s) \tag{5.6}$$

$$\langle\!\langle z, w\rangle\!\rangle_{A(u)\rtimes S_u}(s) = \int_G z(\eta^{-1})^*\alpha_s(w(\eta^{-1}s))d\lambda^u(\eta) \tag{5.7}$$

$${}_{A\rtimes G}\langle\!\langle z, w\rangle\!\rangle(\gamma) = \int_{S_u} \alpha_{\gamma\zeta s}(z(\gamma\zeta s)w(\zeta s)^*)d\beta(s) \tag{5.8}$$

where ζ in (5.8) *is any element of G_u such that $r(\zeta) = s(\gamma)$. In particular, the completion $\mathcal{X}$ of $\mathcal{X}_0$ is an $A \rtimes_\alpha G - A(u) \rtimes_{\alpha|_{S_u}} S_u$-imprimitivity bimodule and $A \rtimes_\alpha G$ is Morita equivalent to $A(u) \rtimes_{\alpha|_{S_u}} S_u$.*

Proof. Suppose (A, G, α) is as in the statement of the theorem and that $u \in G^{(0)}$. First, let $X = G_u$ and recall that X is a (G, S_u)-equivalence with respect to the natural actions of G and S_u on X by Proposition 5.5. Next, consider the trivial bundle $\mathfrak{H} = X \times A(u)$. This is clearly a Banach bundle and the space of compactly supported sections can be identified with $\mathcal{X}_0 = C_c(G_u, A(u))$. Given $\gamma \in X$ we need to equip $\mathfrak{H}_\gamma \cong A(u)$ with an $A(r(\gamma)) - A(u)$-imprimitivity bimodule structure. However $\alpha_\gamma : A(u) \to A(r(\gamma))$ is an isomorphism so that we can use [RW98, Example 3.14] to give $\mathfrak{H}_\gamma$ the bimodule structure with respect to the following operations for $a, b \in A(u)$ and $c \in A(r(\gamma))$:

$$(\gamma, a) \cdot b = (\gamma, ab) \qquad c \cdot (\gamma, a) = (\gamma, \alpha_\gamma^{-1}(c)b)$$

$$\langle(\gamma, a), (\gamma, b)\rangle_{A(u)} = a^*b \qquad {}_{A(r(\gamma))}\langle(\gamma, a), (\gamma, b)\rangle = \alpha_\gamma(ab^*)$$

Next, we define the range and source maps, as well as the G and S_u actions, on $\mathfrak{H}$ in the following manner for $(\gamma, a) \in \mathfrak{H}$, $\eta \in G$ and $s \in S_u$

$$\begin{aligned} r_{\mathfrak{H}}(\gamma, a) &= r(\gamma) & s_{\mathfrak{H}}(\gamma, a) &= u \\ \eta \cdot (\gamma, a) &= (\eta\gamma, a) & (\gamma, a) \cdot s &= (\gamma s, \alpha_s^{-1}(a)). \end{aligned}$$

It is relatively obvious that these are continuous actions. Since $s_{\mathfrak{H}}$ maps onto a single point it must be open. Furthermore, since X is a (G, S_u)-equivalence we know that r_X must be open. However, $r_{\mathfrak{H}}$ is just the composition of r_X with the projection map from $\mathfrak{H}$ onto the first factor and therefore must be open as well. Thus the actions of G and S_u on $\mathfrak{H}$ are strongly continuous in the sense of Definition 1.60. The computation

$$\begin{aligned} (\eta \cdot (\gamma, a)) \cdot s &= (\eta\gamma, a) \cdot s = (\eta\gamma s, \alpha_s^{-1}(a)) \\ &= \eta \cdot (\gamma s, \alpha_s^{-1}(a)) = \eta((\gamma, a) \cdot s) \end{aligned}$$

shows that the actions commute. Now we must verify the different requirements for equivalence. The continuity condition is clear once you recall that multiplication on $\mathcal{A}$ and the action α are both continuous. Furthermore, it is very easy to show that the bundle map $p_{\mathfrak{H}} : \mathfrak{H} \to X$ is equivariant. Next, we must show that the groupoid actions are compatible with the imprimitivity actions. Fix $\gamma \in X$, $a, b \in A(u)$, $c \in A(r(\gamma))$, $s \in S_u$ and $\eta \in G$. We can now perform the following computations without too much difficulty:

$$\begin{aligned} {}_{A(r(\eta))}\langle \eta \cdot (\gamma, a), \eta \cdot (\gamma, b) \rangle &= \alpha_{\eta\gamma}(ab^*) \\ &= \alpha_\eta({}_{A(r(\gamma))}\langle (\gamma, a), (\gamma, b) \rangle), \\ \langle (\gamma, a) \cdot s, (\gamma, b) \cdot s \rangle_{A(u)} &= \langle (\gamma s, \alpha_s^{-1}(a)), (\gamma s, \alpha_s^{-1}(b) \rangle_{A(u)} = \alpha_s^{-1}(a^* b) \\ &= \alpha_s^{-1}(\langle (\gamma, a), (\gamma, b) \rangle_{A(u)}), \\ \eta \cdot (c \cdot (\gamma, a)) &= \eta \cdot (\gamma, \alpha_\gamma^{-1}(c)a) = (\eta\gamma, \alpha_{\eta\gamma}^{-1}(\alpha_\eta(c))a) \\ &= \alpha_\eta(c) \cdot (\eta \cdot (\gamma, a)), \\ ((\gamma, a) \cdot b) \cdot s &= (\gamma s, \alpha_s^{-1}(ab)) = (\gamma s, \alpha_s^{-1}(a)\alpha_s^{-1}(b)) \\ &= ((\gamma, a) \cdot s) \cdot \alpha_s^{-1}(b). \end{aligned}$$

The last thing to check are the invariance conditions. We compute

$$\eta \cdot ((\gamma, a) \cdot b) = (\eta\gamma, ab) = (\eta \cdot (\gamma, a)) \cdot b$$

and

$$(c \cdot (\gamma, a)) \cdot s = (\gamma, \alpha_\gamma^{-1}(c)a) \cdot s = (\gamma s, \alpha_s^{-1}(\alpha_\gamma^{-1}(c)a))$$

$$= (\gamma s, \alpha_{\gamma s}^{-1}(c)\alpha_s^{-1}(a)) = c \cdot (\gamma s, \alpha_s^{-1}(a))$$
$$= c \cdot ((\gamma, a) \cdot s)$$

We have now verified all of the conditions for $\mathfrak{H}$ to be an equivalence between (A, G, α) and $(A(u), S_u, \alpha|_{S_u})$. We use Renault's Equivalence Theorem to conclude that $\mathcal{X}_0$ completes into an $A \rtimes G - A(u) \rtimes S_u$-imprimitivity bimodule. What's more we can use (5.1) through (5.4) to compute the operations on $\mathcal{X}_0$ explicitly. For instance, fix $f \in \Gamma_c(G, r^*\mathcal{A})$, $g \in C_c(S_u, A(u))$ and $z, w \in \mathcal{X}_0$. Observe that if $z(\eta^{-1}\gamma) = (\eta^{-1}\gamma, a)$ then $\eta \cdot z(\eta^{-1}\gamma) = (\gamma, a)$ so that $f(\eta) \cdot (\eta \cdot z(\eta^{-1}\gamma)) = (\gamma, \alpha_\gamma^{-1}(f(\eta))a)$. Making the usual identification of a with $z(\eta^{-1}\gamma)$ we get

$$f \cdot z(\gamma) = \int_G f(\eta) \cdot (\eta \cdot z(\eta^{-1} \cdot \gamma)) d\lambda^{r(\gamma)}(\eta)$$
$$= \int_G \alpha_\gamma^{-1}(f(\eta)) z(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta).$$

In a similar fashion we obtain

$$z \cdot g(\gamma) = \int_{S_u} (z(\gamma \cdot s) \cdot s^{-1}) \cdot \alpha_s(g(s^{-1})) d\beta(s)$$
$$= \int_{S_u} \alpha_s(z(\gamma s) g(s^{-1})) d\beta(s).$$

Next, in (5.3) we may as well pick $x = u$ so that we have

$$\langle\langle z, w\rangle\rangle_{A(u)\rtimes S_u}(s) = \int_G \langle z(\eta^{-1} \cdot u \cdot s^{-1}), w(\eta^{-1} \cdot u) \cdot s^{-1}\rangle_{A(u)} d\lambda^u(\eta)$$
$$= \int_G z(\eta^{-1}s^{-1})^* \alpha_s(w(\eta^{-1})) d\lambda^u(\eta)$$
$$= \int_G z(\eta^{-1})^* \alpha_s(w(\eta^{-1}s)) d\lambda^u(\eta).$$

Finally, given $\gamma \in G$ as in the definition of (5.4) we choose some $\zeta \in X$ such that $r(\zeta) = s(\gamma)$, and then observe

$${}_{A\rtimes G}\langle\langle z, w\rangle\rangle(\gamma) = \int_{S_u} {}_{A(r(\gamma))}\langle z(\gamma \cdot \zeta \cdot s), \gamma \cdot w(\zeta \cdot s)\rangle d\beta(s)$$
$$= \int_{S_u} \alpha_{\gamma\zeta s}(z(\gamma\zeta s) w(\zeta s)^*) d\beta(s)$$

Thus, all of the operations on $\mathcal{X}_0$ have the correct form and we are done. □

Remark 5.8. As we noted in the beginning of the section, equivalence bundles seem frightening because there are many different operations. However, in Theorem 5.7 all of the actions had natural definitions and working with them posed no real difficulty. This is often the case with equivalence bundles.

We also take this opportunity, while we are on the topic of equivalence bundles, to introduce the following technical lemma. It may not seem important but approximate identities like this are very useful for proving nondegeneracy conditions. They can also be quite fiddly which is why we only reference this result.

Remark 5.9. Recall from Definition 1.104 that given a groupoid G a neighborhood of $G^{(0)}$ is called *conditionally compact* if for every compact $K \subset G^{(0)}$ its intersection with $r^{-1}(K)$ is compact.

Lemma 5.10 ([MW08, Proposition 6.7])**.** *Suppose (A, G, α) and (B, H, β) are separable groupoid dynamical systems and $p_{\mathfrak{E}} : \mathfrak{E} \to X$ is an equivalence between (A, G, α) and (B, H, β). Let $\{a_l\}_{l\in\Lambda}$ be an approximate identity for A. Then for each 4-tuple (K, U, l, ϵ) consisting of a compact subset $K \subset G^{(0)}$, a conditionally compact neighborhood U of $G^{(0)}$ in G, $l \in \Lambda$ and $\epsilon > 0$ there is an $e = e_{(K,U,l,\epsilon)} \in \Gamma_c(G, r^*\mathcal{A})$ such that*

(a) $\operatorname{supp} e \subset U$*,*

(b) $\int_G \|e(\gamma)\| d\lambda^u(\gamma) \leq 4$ *for all* $u \in K$ *and*

(c) $\left\| \int_G e(\gamma) d\lambda^u(\gamma) - a_l(u) \right\| < \epsilon$ *for all* $u \in K$*.*

Furthermore $\{e_{(K,U,l,\epsilon)}\}$ directed by increasing K and l and decreasing U and ϵ is an approximate identity with respect to the inductive limit topology for both the left action of $\Gamma_c(G, r^*\mathcal{A})$ on itself and the left action of $\Gamma_c(G, r^*\mathcal{A})$ on $\Gamma_c(X, \mathfrak{E})$. Specifically, $e_{(K,U,l,\epsilon)} * g \to g$ with respect to the inductive limit topology for all $g \in \Gamma_c(G, r^*\mathcal{A})$ and $e_{(K,U,l,\epsilon)} \cdot z \to z$ with respect to the inductive limit topology for all $z \in \Gamma_c(X, \mathfrak{E})$.

Remark. This is just an explanation of how to extract this statement of the lemma from the reference. Those readers not concerned with chasing down citations can skip ahead. Lemma 5.10 is really a result of both [MW08, Lemma 6.6, Lemma 6.7]. Specifically, [MW08, Lemma 6.6] states that the net $\{e_{(K,l,U,\epsilon)}\}$ given above is an approximate identity in the inductive limit topology. Then such a net is constructed in the proof of [MW08, Lemma 6.7]. The construction of this approximate identity is fairly complex and occupies the whole of [MW08, Section 6]. $\square$

5.2 Invariant Ideals

This is mainly a technical section. We will investigate certain nice ideals of crossed products and show that they behave in a reasonable manner. This material is structured along the lines of [Wil07, Section 3.4]. We start by considering what happens when we cut down upper-semicontinuous bundles.

Definition 5.11. Let A be a $C_0(X)$-algebra, $\mathcal{A}$ its associated upper-semicontinuous bundle, and Y a locally compact subset of X. We define the *restriction* of A to Y to be

$$A(Y) := \Gamma_0(Y, \mathcal{A}).$$

Proposition 5.12. *Suppose A is a $C_0(X)$-algebra, $\mathcal{A}$ its associated upper-semicontinuous bundle, and Y is a nonempty locally compact subset of X. Then $A(Y)$ is a $C_0(Y)$-algebra with $A(Y)(y) = A(y)$ for all $y \in Y$. Furthermore the upper-semicontinuous bundle associated to $A(Y)$ is $\mathcal{A}|_Y = p^{-1}(Y)$.*

Proof. Let $\mathcal{B} = p^{-1}(Y)$. It follows immediately from the fact that $\mathcal{A}$ is an upper-semicontinuous C^*-bundle that $\mathcal{B}$ is as well. Observe that, by definition, $A(Y)$ is the space of sections of $\mathcal{B}$ which vanish at infinity. It follows from Example 3.18 that $A(Y)$ is a $C_0(Y)$-algebra. Now, once we make the usual identification of $A(Y)(y)$ with the fibre $\mathcal{B}_y$, we have $A(Y)(y) = \mathcal{B}_y = \mathcal{A}_y = A(y)$. Finally, still using this identification, it is vacuously true that $\mathcal{B}$ has the unique topology making $A(Y)$ into the space of sections of $\mathcal{B}$ which vanish at infinity. Thus, by definition, $\mathcal{B}$ is the upper-semicontinuous bundle associated to $A(Y)$. □

Now, the notation $A(Y)$ is going to be subject to the same abuse that we subject $a(x)$ to. Specifically, the next proposition shows that for closed sets $A(Y)$ is (isomorphic to) a quotient of A, bringing us back to definition of the similarly denoted $A(x)$.

Remark 5.13. In the same vein as Definition 3.15, if C is a closed subset of X and J_C is the ideal of functions in $C_0(X)$ which vanish on C then we will denote the ideal $\overline{\Phi_A(J_C) \cdot A}$ by I_C.

Proposition 5.14. *Suppose A is a $C_0(X)$-algebra and $\mathcal{A}$ is its associated upper-semicontinuous bundle. Let U be an open subset of X and $C = X \setminus U$. Then there are $*$-homomorphisms $\iota : A(U) \to A$ and $\rho : A \to A(C)$ where ι is the inclusion of $\Gamma_0(U, \mathcal{A})$ into $\Gamma_0(X, \mathcal{A})$ and ρ is the restriction map from $\Gamma_0(X, \mathcal{A})$ into $\Gamma_0(C, \mathcal{A})$. Furthermore, the sequence*

$$0 \longrightarrow A(U) \xrightarrow{\iota} A \xrightarrow{\rho} A(C) \longrightarrow 0$$

is short exact. Finally, $\operatorname{ran} \iota = I_C$ *so that $A(U)$ is isomorphic to I_C and $A(C)$ is isomorphic to the quotient A/I_C.*

Proof. We will identify A with the sections of $\mathcal{A}$ which vanish at infinity. Recall that, by definition, $A(U)$ and $A(C)$ are the sections on U and C which vanish at infinity. Our first goal is to see that ι and ρ are well defined. Suppose $f \in A(U)$ and let $\iota(f)$ be the extension of f to X by letting $\iota(f)(x) = 0_x$ for all $x \notin U$. We claim that $\iota(f)$ is continuous on X. Suppose $x_i \to x$ and $x \in U$. Then, eventually, $x_i \in U$ so that $\iota(f)(x_i) = f(x_i) \to f(x) = \iota(f)(x)$. Now suppose $x \notin U$. As we have done before, we will pass to a subnet and show that a sub-subnet converges to $\iota(f)(x) = 0_x$. If $x_i \notin U$ infinitely often then pass to a subnet and assume this is always true. It follows from condition (d) of Definition 3.1 that $\iota(f)(x_i) = 0_{x_i} \to 0_x$. If $x_i \in U$ eventually then pass to a subnet and assume this is always true. If $\epsilon > 0$ then, since f vanishes at infinity, $K = \{x \in U : \|f(x)\| \geq \epsilon\}$ is a compact subset of U and is therefore closed in X. Thus, $X \setminus K$ is an open neighborhood of x and therefore eventually contains x_i. However, this implies that eventually $\|f(x_i)\| < \epsilon$. Since $\epsilon > 0$ was arbitrary we must have $\|f(x_i)\| \to 0$ and it follows that $f(x_i) \to 0_x$. Thus $\iota(f)$ is continuous. This also shows that $\iota(f)$ vanishes at infinity in X, since the set K above is compact as a subset of X. Finally, it is clear that ι is a $*$-homomorphism under the pointwise operations and that ι is isometric with respect to the uniform norm. It follows that ι is injective and an isomorphism onto its range.

Clearly $\rho(f)$ is a continuous section on C. Furthermore if $K = \{x : \|f(x)\| \geq \epsilon\}$ for some $\epsilon > 0$ then $K \cap C$ is a closed subset of a compact set and is therefore compact. It follows that $\rho(f)$ vanishes at infinity. Obviously, ρ is a $*$-homomorphism with respect to the pointwise operations and $\|\rho(f)\|_\infty \leq \|f\|_\infty$. Now consider $\operatorname{ran}\rho$. We would like to show that $\operatorname{ran}\rho$ is closed under the $C_0(C)$-action. Well, $\operatorname{ran}\rho$ is closed and if $\phi_i \to \phi$ uniformly in $C_0(C)$ then $\phi_i \cdot a \to \phi \cdot a$ uniformly in $A(C)$. It follows that it suffices to show that $\operatorname{ran}\rho$ is closed under the action of $C_c(C)$. Given $\phi \in C_c(C)$ use Lemma 1.71 to extend ϕ to $C_c(X)$. We then observe that $\rho(\phi \cdot f) = \phi \cdot \rho(f)$ for all $f \in A$. Hence $\operatorname{ran}\rho$ is closed under the action of $C_0(C)$. Now, given $x \in C$ and $a \in A(C)(x) = A(x)$ we pick $f \in A$ such that $f(x) = a$. Then $\rho(f)(x) = f(x) = a$. Thus $\operatorname{ran}\rho$ is fibrewise dense. It follows from Proposition 3.11 that $\operatorname{ran}\rho$ is dense in $A(C)$. Since $\operatorname{ran}\rho$ is closed it follows that ρ is surjective.

All we have to do now is show that $\operatorname{ran}\iota = \ker\rho = I_C$. If $f \in A(U)$ then $\rho(\iota(f))(x) = \iota(f)(x) = 0_x$ for all $x \in C = X \setminus U$. Thus $\operatorname{ran}\iota \subset \ker\rho$. Now suppose that $f \in \ker\rho$ so that f is zero on C. (Note: The following argument is slightly easier if we are in the second countable case.) Given $\epsilon > 0$ we know that $K = \{x \in X : \|f(x)\| \geq \epsilon/2\}$ is compact. Using the fact that K is a compact set which is disjoint from C we can find a relatively compact open neighborhood U of K which is disjoint from C. Use Urysohn's Lemma to find a function $\phi \in C_c(\overline{U})$ such that ϕ is one K and zero on $\overline{U} \setminus U$. Now extend ϕ to $\overline{U} \cup C$ by letting ϕ be zero on C. If $x_i \to x$ and $x \in U$ then x_i is eventually in U and $\phi(x_i) \to \phi(x)$ by construction. Suppose $x_i \to x$ and $x \notin U$. Then if $x_i \notin U$ infinitely often pass to

a subnet and assume this is always true. It follows that $\phi(x_i) = 0 \to 0 = \phi(x)$. If eventually $x_i \in U$ pass to a subnet and assume that this is always true. But then $x \in \overline{U}$ and $\phi(x_i) \to \phi(x)$ by assumption. Thus, our extension ϕ is a continuous, compactly supported function on $C \cup \overline{U}$. We can now use Lemma 1.71 to extend ϕ to a function in $C_c(X)$. Since ϕ was constructed to be zero on C we have $\phi \cdot f \in I_C$. Furthermore it is easy to check that $\|\phi \cdot f - f\|_\infty < \epsilon$. Since $\epsilon > 0$ was arbitrary it follows that $f \in I_C$ and thus $\ker \rho \subset I_C$. Finally, given $\phi \in C_c(X)$ such that ϕ is zero on C and $f \in A$ observe that $\phi \cdot f$ is zero on C. Hence, if $\epsilon > 0$ then $K = \{x \in X : \|\phi \cdot f(x)\| \geq \epsilon\} \subset U$. It follows that the restriction of $\phi \cdot f$ to U, denoted g, vanishes at infinity. Since $\iota(g) = \phi \cdot f$ it follows that $I_C \subset \operatorname{ran} \iota$. Thus $\operatorname{ran} \iota = \ker \rho = I_C$ and the rest of the proposition follows. □

As a corollary to the above we find that the spectra of these "restriction" bundles are well behaved.

Corollary 5.15. *Suppose A is a $C_0(X)$-algebra, U is an open subset of X and $C = X \setminus U$. Let $\sigma : \widehat{A} \to X$ be the associated map on the spectrum. Then $\sigma^{-1}(U) \cong (A(U))^\wedge$ and $\sigma^{-1}(C) \cong (A(C))^\wedge$. Furthermore, if we view both $A(U)$ and $\sigma^{-1}(U)$ as the disjoint union $\coprod_{x \in U} A(x)^\wedge$ then this identification is given by the identity. The corresponding statement holds for C.*

Proof. The map ι is an isomorphism of $A(U)$ onto I_C and as such we may identify $\widehat{I_C}$ and $(A(U))^\wedge$ via the map $\phi_1(\pi) = \pi \circ \iota$. Furthermore, since I_C is an ideal in A we can, and do, identify the set $\{\pi \in \widehat{A} : I_C \not\subset \ker \pi\}$ with $\widehat{I_C}$ [RW98, Proposition A.26] via the map $\phi_2(\pi) = \pi|_{I_C}$. We would like to show that $\widehat{I_C} = \sigma^{-1}(U)$. If $\pi \in \sigma^{-1}(U)$ then, by definition, $I_x \subset \ker \pi$ for some $x \in U$ so that π factors to an irreducible representation π' of $A(x)$. Since π' is irreducible it must be non-zero, so there exists $a \in A(x)$ such that $\pi(x) \neq 0$. However, if we choose $b \in A$ such that $b(x) = a$ and $\phi \in C_c(U)$ such that $\phi(x) = 1$ then $\phi \cdot b \in I_C$ and

$$\pi(\phi \cdot b) = \pi'(\phi(x)b(x)) = \pi(a) \neq 0.$$

Thus $\pi \in \widehat{I_C}$. Next suppose $\pi \in \widehat{I_C}$. By construction, $I_{\sigma(\pi)} \subset \ker \pi$. If $\sigma(\pi) \in C$ then $I_C \subset I_{\sigma(\pi)} \subset \ker \pi$ but this is a clear contradiction. At this point we can form the homeomorphism $\phi = \phi_2 \circ \phi_1$ mapping $\sigma^{-1}(U)$ onto $A(U)$ defined via $\phi(\pi) = \pi|_{I_C} \circ \iota$. Fix $x \in U$ and suppose $\pi \in A(x)^\wedge$. Let π' be the lift of π to A and π'' the lift of π to $A(U)$. Observe that

$$\phi(\pi')(a) = \pi'(\iota(a)) = \pi(\iota(a)(x)) = \pi(a(x)) = \pi''(a).$$

It follows that if we identify both $A(U)$ and $\sigma^{-1}(U)$ with $\coprod_{x \in U} A(x)^\wedge$ then ϕ is given by the identity map.

Next, observe that the restriction map ρ factors to an isomorphism $\bar{\rho}$ of A/I_C onto $A(C)$ and thus we can define a homeomorphism from $(A/I_C)^\wedge$ onto $(A(C))^\wedge$ by $\phi_1(\pi) = \pi \circ \bar{\rho}^{-1}$. Furthermore we can, and do, identify the set $\{\pi \in \widehat{A} : I_C \subset \ker \pi\}$ with $(A/I_C)^\wedge$ [RW98, Proposition A.28] via the map $\phi_2(\pi) = \bar{\pi}$ where $\bar{\pi}$ is the factorization of π to A/I_C. We would like to show that $(A/I_C)^\wedge = \sigma^{-1}(C)$. Suppose $\pi \in \widehat{A}$ such that $I_C \subset \ker \pi$ and suppose $x = \sigma(\pi) \notin C$. By definition π factors to an irreducible representation π' of $A(x)$ and therefore there exists $a \in A(x)$ such that $\pi'(a) \neq 0$. Now find $b \in A$ such that $b(x) = a$ and $\phi \in C_c(U)$ such that $\phi(x) = 1$. Then $\phi \cdot b \in I_C$ so that

$$0 = \pi(\phi \cdot b) = \pi'(\phi(x)b(x)) = \pi'(a)$$

which is a contradiction. Now suppose $\pi \in \widehat{A}$ such that $\sigma(\pi) \in C$. Then $I_C \subset I_x \subset \pi$ and we are done. Thus we can define a homeomorphism from $\sigma^{-1}(C)$ onto $A(C)^\wedge$ via $\phi = \phi_1 \circ \phi_2$. Furthermore given $\pi \in \sigma^{-1}(C)$ we have $\phi(\pi) = \bar{\pi} \circ \bar{\rho}^{-1}$. Now fix $x \in C$ and suppose $\pi \in A(x)^\wedge$. Let π' be the lift of π to A and π'' the lift of π to $A(C)$. If $q : A \to A/I_C$ is the quotient map and $a \in A$ then

$$\phi(\pi')(\bar{\rho}(q(a))) = \overline{\pi'}(q(a)) = \pi'(a) = \pi(a(x)) = \pi(\rho(a)(x)) = \pi''(\rho(a)) = \pi''(\bar{\rho}(q(a))).$$

Since $\bar{\rho} \circ q$ is surjective this implies that $\phi(\pi') = \pi''$. Thus we can view ϕ as the identity map. □

Now we will extend some of this theory to groupoid crossed products. In particular we need to be able to cut down groupoids by restricting the unit space.

Definition 5.16. Suppose G is a locally compact Hausdorff groupoid and Y is a locally compact G-invariant subset of $G^{(0)}$. Then we define the *restriction* of G to Y to be $G|_Y := r^{-1}(Y) = s^{-1}(Y)$.

Proposition 5.17. *Let G be a locally compact Hausdorff groupoid with Haar system λ and Y a locally compact G-invariant subset of $G^{(0)}$. Then $G|_Y$ is a locally compact Hausdorff subgroupoid of G. Furthermore the restriction of the Haar system to $G|_Y$ is a Haar system. We will always equip $G|_Y$ with this Haar system.*

Proof. Since Y is locally compact in $G^{(0)}$ it must be the intersection of an open set and a closed set [Wil07, Lemma 1.25,1.26]. It follows that $G|_Y$ is locally compact and, since Y is G-invariant, that $G|_Y$ is a subgroupoid of G such that $(G|_Y)^u = G^u$ for all $u \in Y$. Consider the restriction of the Haar system to $G|_Y$. Because $G|_Y$ is the restriction of G to a G-invariant subset, $\operatorname{supp} \lambda^u = (G|_Y)^u = G^u$ for all $u \in Y$. Given a compactly supported function $f \in C_c(G|_Y)$ we can extend f to G using Lemma 1.71 and then the continuity of the Haar system follows. Finally, left-invariance is immediate from the fact that λ is a Haar system for G. □

Of course, we would like to be able to restrict a groupoid action to $G|_Y$, but this means restricting A as well. The result is the following

Proposition 5.18. *Suppose (A, G, α) is a groupoid dynamical system and Y is a locally compact G-invariant subset of $G^{(0)}$. Then the restriction of α to $G|_Y$ defines an action of $G|_Y$ on $A(Y)$.*

Proof. The fact that $G|_Y$ is a groupoid with a Haar system follows from Proposition 5.17 since Y is G-invariant. Since $A(Y)(y) = A(y)$, it is easy to see that α satisfies the first two conditions of an action of $G|_Y$ on $A(Y)$. On the other hand, since the bundle associated to $A(Y)$ is just the restriction of the bundle associated to A, the continuity condition is easy to verify as well. □

Now that we can restrict actions we will show, similar to Proposition 5.14, that the crossed product respects the restrictions. In particular, we will have to work with the following object.

Definition 5.19. Given a separable groupoid dynamical system (A, G, α) and an open G-invariant subset U of $G^{(0)}$ let $\mathrm{Ex}(U)$ be the closure in $A \rtimes_\alpha G$ of the set

$$\{f \in \Gamma_c(G, r^*\mathcal{A}) : \operatorname{supp} f \subset G|_U\}.$$

There are more elementary ways to prove the next result, however the following proof, inspired by [Cla04, Lemma 3.3.1], is pretty slick.

Proposition 5.20. *Given a separable groupoid dynamical system (A, G, α) and an open G-invariant subset U of $G^{(0)}$ then the inclusion map*

$$\iota : \Gamma_c(G|_U, r^*\mathcal{A}) \to \Gamma_c(G, r^*\mathcal{A})$$

extends to an isomorphism of $A(U) \rtimes_\alpha G|_U$ onto $\mathrm{Ex}(U)$. Furthermore, $\mathrm{Ex}(U)$ is an ideal in $A \rtimes G$.

Proof. First, observe that $G|_U$ is an open subset of G so that we can apply Proposition 5.14 to r^*A and see that $\iota(f)$ is continuous for all $f \in \Gamma_c(G|_U, r^*\mathcal{A})$. However, $\operatorname{supp} \iota(f) = \operatorname{supp} f$ and it follows that $\iota(f) \in \mathrm{Ex}(U)$. Since the action of $G|_U$ on $A(U)$ is just the restriction of the action of G on A, it is straightforward to show that ι is a $*$-homomorphism. Furthermore, it is clear that ι is continuous with respect to the inductive limit topology. Next, observe that if $f, g \in \Gamma_c(G, r^*\mathcal{A})$ such that $\operatorname{supp} f \subset G|_U$ then for all $\gamma \notin G|_U$ we have

$$f * g(\gamma) = \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) = 0 \tag{5.9}$$

since $r(\eta) = r(\gamma) \in U$ implies $\eta \in G|_U$. Similarly we find that $g * f(\gamma) = 0$ for all $\gamma \notin G|_U$. This shows that $g * f, f * g \in \mathrm{Ex}(U)$ and it is enough to imply that $\mathrm{Ex}(U)$ is an ideal in $A \rtimes G$. Furthermore, if $f \in \Gamma_c(G, r^*\mathcal{A})$ such that $\operatorname{supp} f \subset G|_U$ then we can clearly view f as a compactly supported section on $G|_U$ and in this case $\iota(f) = f$. Thus ι maps onto a dense subset of $\mathrm{Ex}(U)$.

Now, it follows from Corollary 3.134 that ι is bounded and extends to a $*$-homomorphism on $A(U) \rtimes G|_U$. Furthermore, it is clear that $\operatorname{ran} \iota = \mathrm{Ex}(U)$. Next, suppose R is a faithful representation of $A(U) \rtimes G|_U$ on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{H}_0 = \operatorname{span}\{R(f)h : f \in \Gamma_c(G|_U, r^*\mathcal{A}), h \in \mathcal{H}\}$ and observe that $\mathcal{H}_0$ is dense in $\mathcal{H}$. If $f \in \operatorname{ran} \iota$ and $g \in \Gamma_c(G, r^*\mathcal{A})$ then we know from (5.9) that $f * g(\gamma) = 0$ unless $r(\gamma) \in r(\operatorname{supp} f) \subset U$. Thus $r(\operatorname{supp} f * g) \subset U$ so that $\operatorname{supp} f * g \subset G|_U$. In particular, we can view $f*g$ as a function in $\Gamma_c(G|_U, r^*\mathcal{A})$. We define a representation of $\Gamma_c(G, r^*\mathcal{A})$ on $\mathcal{H}_0$ via

$$T(f) \sum_{i=1}^n R(g_i)h_i = \sum_{i=1}^n R(f * g_i)h_i. \tag{5.10}$$

Of course, we need to check that T is well defined. It will suffice to show that if $\sum_i R(g_i)h_i = 0$ then $T(f) = 0$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$. Let $\{e_\kappa\} \subset \Gamma_c(G|_U, r^*\mathcal{A})$ be the approximate identity from Lemma 5.10 so that $e_\kappa * g_i \to g_i$ with respect to the inductive limit topology for all i. We now have,

$$\begin{aligned}
\sum_{i=1}^n R(f * g_i)h_i &= \sum_{i=1}^n R(f * \lim_\kappa (e_\kappa * g_i)) \\
&= \sum_{i=1}^n R(\lim_\kappa (f * e_\kappa) * g_i)h_i \\
&= \lim_\kappa R(f * e_\kappa) \sum_{i=1}^n R(g_i)h_i = 0
\end{aligned}$$

where each limit, except for the last, is taken in the inductive limit topology. Thus T is well defined and it is easy to see that it is a homomorphism into the algebra of linear operators on $\mathcal{H}_0$. Next, observe that if $f \in \Gamma_c(G|_U, r^*\mathcal{A})$ then $T(\iota(f)) = R(f)$. We now verify the conditions of Theorem 3.119. First, we have

$$T(\iota(e_k))R(f)h = R(e_k * f)h \to R(f)h$$

for all $f \in \Gamma_c(G|_U, r^*\mathcal{A})$ and $h \in \mathcal{H}$. Since R is nondegenerate this suffices to show that the set $\operatorname{span}\{T(f)k : f \in \Gamma_c(G, r^*\mathcal{A}), k \in \mathcal{H}_0\}$ is dense in $\mathcal{H}$. In order to verify

the continuity condition it clearly suffices to show that

$$f \mapsto (T(f)R(g)h, R(k)l)$$

is continuous with respect to the inductive limit topology for all $g, k \in \Gamma_c(G|_U, r^*\mathcal{A})$ and $h, l \in \mathcal{H}$. However, $(T(f)R(g), R(k)l) = (R(f * g)h, R(k)l)$ so the continuity follows from the fact that R and convolution are both continuous with respect to the inductive limit topology. Lastly, for the third condition it will suffice to check that

$$(T(f)R(g)h, R(k)l) = (R(g)h, T(f^*)R(k)l)$$

for all g, h and k, l as before. We can compute

$$\begin{aligned}(T(f)R(g)h, R(k)l) &= (R(f * g)h, R(k)l) = (h, R((f * g)^* * k)l) \\ &= (h, R(g^* * f^* * k)l) = (R(g)h, R(f^* * k)l) \\ &= (R(g), T(f^*)R(k)l).\end{aligned}$$

Notice that we are being a little schizophrenic about which algebra the convolution and involution are actually occurring in. However, since the Haar system and action of $(A(U), G, \alpha|_U)$ are the restrictions of the Haar system and action of (A, G, α), the convolution and involution formulas for both algebras are the same. In any case, by Theorem 3.119, T is bounded with respect to the universal norm and extends to a representation of $A \rtimes G$. Furthermore, since $R = T \circ \iota$ on $\Gamma_c(G|_U, r^*\mathcal{A})$ this identity holds in general. Thus, given $f \in A(U) \rtimes G|_U$ we have

$$\|f\| = \|R(f)\| = \|T(\iota(f))\| \leq \|\iota(f)\|.$$

It follows that ι is isometric and we are done. □

The complement to Proposition 5.20 is the following

Proposition 5.21. *Suppose (A, G, α) is a separable groupoid dynamical system and C is a closed G-invariant subset of $G^{(0)}$. Then the restriction map*

$$\rho : \Gamma_c(G, r^*\mathcal{A}) \to \Gamma_c(G|_C, r^*\mathcal{A})$$

extends to a surjective homomorphism from $A \rtimes G$ onto $A(C) \rtimes G|_C$. Furthermore, $\rho(\Gamma_c(G, r^*\mathcal{A}))$ is dense in $\Gamma_c(G|_C, r^*\mathcal{A})$ with respect to the inductive limit topology.

Proof. Since C is closed, $\rho(f)$ is compactly supported in $G|_C$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$. Thus ρ is well defined and it is straightforward to see that it is a $*$-homomorphism which is continuous with respect to the inductive limit topology. It follows from Proposition 3.133 that ρ extends to a $*$-homomorphism on $A \rtimes G$. It follows from

Proposition 5.14 that the restriction map from $C_0(G)$ onto $C_0(G|_C)$ is surjective. Given $\phi \in C_0(G|_C)$ use the aforementioned surjectivity to extend ϕ to a function $\tilde{\phi} \in C_0(G)$ and observe that $\phi \cdot \rho(f) = \rho(\tilde{\phi} \cdot f)$. Thus, $\operatorname{ran} \rho$ is closed under the action of $C_0(G|_C)$. Next, given $\gamma \in G|_C$ and $a \in A(r(\gamma))$ choose $b \in A$ such that $b(r(\gamma)) = a$ and $f \in C_c(G)$ such that $f(\gamma) = 1$. Then $\rho(f \cdot b)(\gamma) = f(\gamma)b(r(\gamma)) = a$. Thus $\operatorname{ran} \rho$ is fibrewise dense and therefore dense in the uniform norm by Proposition 3.11. What's more, we can perform the standard trick of multiplying a uniformly converging sequence by an appropriately supported function in $C_c(G)$ to see that $\operatorname{ran} \rho$ is dense in the inductive limit topology. Hence $\operatorname{ran} \rho$ is dense in $A(C) \rtimes G|_C$ and therefore ρ is surjective. □

Now we can put everything together to get a nice result mimicking [Wil07, Proposition 3.19].

Theorem 5.22. *Let (A, G, α) be a separable groupoid dynamical system, U an open G-invariant subset of $G^{(0)}$, and C the closed G-invariant set $G^{(0)} \setminus U$. Then inclusion and restriction extend to $*$-homomorphisms $\iota : A(U) \rtimes G|_U \to A \rtimes G$ and $\rho : A \rtimes G \to A(C) \rtimes G|_C$, respectively. Furthermore, the following sequence is short exact*

$$0 \longrightarrow A(U) \rtimes G|_U \xrightarrow{\ \iota\ } A \rtimes G \xrightarrow{\ \rho\ } A(C) \rtimes G|_C \longrightarrow 0$$

and $\operatorname{ran} \iota = \ker \rho = \operatorname{Ex}(U)$ so that $A(C) \rtimes G|_C$ is isomorphic to the quotient space $A \rtimes G / \operatorname{Ex}(U)$.

Proof. Proposition 5.20 shows that ι is well defined and injective, and Proposition 5.21 shows that ρ is well defined and surjective. All that is left is to show that $\operatorname{ran} \iota = \ker \rho = \operatorname{Ex}(U)$. We have already shown that $\operatorname{ran} \iota = \operatorname{Ex}(U)$. Furthermore, it is clear that given $f \in \Gamma_c(G|_U, r^*\mathcal{A})$ we have $\rho \circ \iota(f) = 0$. It follows that $\operatorname{ran} \iota \subset \ker \rho$. Thus we are reduced to proving that $\ker \rho \subset \operatorname{Ex}(U)$.

Let R be a representation of $A \rtimes G$ such that $\ker R = \operatorname{Ex}(U)$. Now, suppose $f, g \in \Gamma_c(G, r^*\mathcal{A})$ such that $\rho(f) = \rho(g)$. Unfortunately, just because $f - g$ is zero on $G|_C$ doesn't mean $f - g$ is supported on $G|_U$. However, consider $K = \{\gamma \in G : \|f(x) - g(x)\| \geq \epsilon\}$. Since K is a closed subset of $\operatorname{supp}(f - g)$ it must be compact. Furthermore, since K is compact and disjoint from $G|_C$ we can find some relatively compact neighborhood V of K such that $K \subset V \subset G|_U$. Now choose $\phi \in C_c(G)$ such that ϕ is one on K and zero off V. Then $\phi \cdot (f - g)$ is supported inside $G|_U$ so that $\phi \cdot (f - g) \in \operatorname{Ex}(U)$. However, it is easy to see that $\|\phi \cdot (f - g) - (f - g)\|_\infty < \epsilon$. Since $\operatorname{supp} \phi \cdot (f - g) \subset \operatorname{supp} f - g$ we can use this construction to find a sequence in $\operatorname{Ex}(U)$ which converges to $f - g$ in the inductive limit topology. It follows that $f - g \in \operatorname{Ex}(U) = \ker R$. Thus the representation T of $\Gamma_c(G|_C, r^*\mathcal{A})$ given by

$$T(\rho(f)) = R(f)$$

is well defined. Furthermore, since R and ρ are $*$-homomorphisms, it follows that T is as well. We would like to see that T is I-norm decreasing. Suppose $f \in \Gamma_c(G, r^*\mathcal{A})$ and fix $\epsilon > 0$. Since Lemma 4.32 implies that $u \mapsto \int_G \|f(\gamma)\| d\lambda^u(\gamma)$ is upper-semicontinuous we can find for each $v \in r(\operatorname{supp} f) \cap C$ some relatively compact open set O_v such that $w \in O_v$ implies

$$\int_G \|f(\gamma)\| d\lambda^w(\gamma) \leq \int_G \|f(\gamma)\| d\lambda^v(\gamma) + \epsilon \leq \|\rho(f)\|_I + \epsilon. \tag{5.11}$$

By considering the continuous compactly supported function $\gamma \mapsto f(\gamma^{-1})$ we can, in the same fashion, also find for each $v \in r(\operatorname{supp} f) \cap C$ some relatively compact open set V_v such that $w \in V_v$ implies

$$\int_G \|f(\gamma)\| d\lambda_w(\gamma) \leq \int_G \|f(\gamma)\| d\lambda_v(\gamma) + \epsilon \leq \|\rho(f)\|_I + \epsilon. \tag{5.12}$$

Since $\{O_v \cap V_v\}$ is an open cover of the compact set $r(\operatorname{supp} f) \cap C$, there exists some finite subcover $\{O_{v_i} \cap V_{v_i}\}$. Let $O = \bigcup_i O_{v_i} \cap V_{v_i}$ and observe that, because the union is finite, O is still relatively compact. Now choose $\phi \in C_c(G^{(0)})$ such that ϕ is one on $r(\operatorname{supp} f) \cap C$, zero off O, and $0 \leq \phi \leq 1$, and define $g \in \Gamma_c(G, r^*\mathcal{A})$ by $g(\gamma) = \phi(r(\gamma)) f(\gamma)$. If $v \in O$ then, by construction,

$$\begin{aligned} \phi(v) \int_G \|f(\gamma)\| d\lambda^v(\gamma) &\leq \|\rho(f)\|_I + \epsilon, \quad \text{and} \\ \phi(v) \int_G \|f(\gamma)\| d\lambda_v(\gamma) &\leq \|\rho(f)\|_I + \epsilon. \end{aligned}$$

Furthermore, if $v \notin O$ then

$$\begin{aligned} \phi(v) \int_G \|f(\gamma)\| d\lambda^v(\gamma) &= 0 \leq \|\rho(f)\|_I + \epsilon, \quad \text{and} \\ \phi(v) \int_G \|f(\gamma)\| d\lambda_v(\gamma) &= 0 \leq \|\rho(f)\|_I + \epsilon. \end{aligned}$$

It follows that $\|g\|_I \leq \|\rho(f)\|_I + \epsilon$. However, $g - f$ is zero on C by construction so that

$$\|T(\rho(f))\| = \|R(f)\| = \|R(g)\| \leq \|g\| \leq \|g\|_I \leq \|\rho(f)\|_I + \epsilon.$$

Since ϵ was chosen arbitrarily, this implies that T is I-norm decreasing. Since T is an I-norm decreasing $*$-representation, it follows that T extends to a representation of $A(C) \rtimes G|_C$. Finally, since the identity $T \circ \rho = R$ holds on a dense subset it holds everywhere. Thus $\ker \rho \subset \ker R = \operatorname{Ex}(U)$ and we are done. □

Remark 5.23. This section serves as an excellent demonstration of the fact that kernels are not well behaved with respect to completions. In Proposition 5.17 it is clear that ι is injective on a dense subalgebra but this does not imply that its extension to the crossed product is injective. We had to put in the extra effort to show that it was isometric. In Proposition 5.21 it is easy to show that those elements in $\Gamma_c(G, r^*\mathcal{A})$ for which $\rho(f) = 0$ are contained in $\operatorname{Ex}(U)$. However, $\ker \rho$ is *not* the completion of $\ker \rho \cap \Gamma_c(G, r^*\mathcal{A})$. The solution in this case was to work with representations since they are determined by their action $\Gamma_c(G, r^*\mathcal{A})$.

Of course, if we restrict Theorem 5.22 to the group bundle case then there is even more structure to worry about. In particular, we want to see that restriction preserves the bundle structure of the crossed product.

Proposition 5.24. *Suppose (A, S, α) is a separable dynamical system, S is a group bundle, and that U is an open subset of $S^{(0)}$. Then $A \rtimes_\alpha S(U)$ and $A(U) \rtimes_\alpha S|_U$ are isomorphic as $C_0(U)$-algebras. Similarly if C is a closed subset of $S^{(0)}$ then $A \rtimes_\alpha S(C)$ and $A(C) \rtimes_\alpha S|_C$ are isomorphic as $C_0(C)$-algebras.*

Proof. Let U be open in $S^{(0)}$ and let $C = S^{(0)} \setminus U$. Recall that $A \rtimes S$ is a $C_0(S^{(0)})$-algebra. It follows from Proposition 5.14 that $A \rtimes S(U)$ is isomorphic to the ideal

$$\begin{aligned} I_U &= \overline{\operatorname{span}}\{\phi \cdot f : \phi \in C_0(S^{(0)}), f \in A \rtimes S, \phi(C) = 0\} \\ &= \overline{\operatorname{span}}\{\phi \cdot f : \phi \in C_0(S^{(0)}), f \in \Gamma_c(S, p^*\mathcal{A}), \phi(C) = 0\} \end{aligned}$$

via the inclusion map $\iota_1 : A \rtimes S(U) \to A \rtimes S$. We claim that $I_U = \operatorname{Ex}(U)$. Recall that $\operatorname{Ex}(U)$ is the closure of $\Gamma_c(S|_U, p^*\mathcal{A})$ inside $A \rtimes S$. Given $\phi \in C_0(G^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{A})$ such that $\phi(C) = 0$ we would like to show $\phi \cdot f \in \operatorname{Ex}(U)$. Let $K = \{v \in S^{(0)} : |\phi(v)| \geq \epsilon\}$ and observe that K is disjoint from C. Thus we can find a function $\psi \in C_c(S^{(0)})^+$ such that ψ is one on K and ψ is zero off a neighborhood $V \subset U$ of K. Then $\operatorname{supp} \psi\phi \cdot f \subset p^{-1}(V) \subset S|_U$ so that $\psi\phi \cdot f \in \operatorname{Ex}(U)$. Furthermore we constructed ψ so that $\|\psi\phi \cdot f - \phi \cdot f\|_\infty < \epsilon$. Since we also have $\operatorname{supp} \psi\phi \cdot f \subset \operatorname{supp} f$ we can use this construction to find a sequence in $\operatorname{Ex}(U)$ which converges to $\phi \cdot f$ in the inductive limit topology. Hence $\phi \cdot f \in \operatorname{Ex}(U)$ and it follows that $I_C \subset \operatorname{Ex}(U)$. Next suppose $f \in \Gamma_c(S|_U, p^*\mathcal{A})$. Then $\operatorname{supp} f$ is a compact set which is disjoint from C. Let $\phi \in C_c(S^{(0)})^+$ be one on K and zero on C. Then $f = \phi \cdot f \in I_C$ and it follows that $\operatorname{Ex}(U) \subset I_C$. Now, we also know that the inclusion map $\iota_2 : \Gamma_c(S|_U, p^*\mathcal{A}) \to \Gamma_c(S, p^*\mathcal{A})$ extends to an isomorphism of $A(U) \rtimes S|_U$ with $\operatorname{Ex}(U)$. Consider the isomorphism $\iota_2^{-1} \circ \iota_1$ from $A(U) \rtimes S|_U$ onto $A \rtimes S(U)$. If $\phi \in C_0(U)$ and $f \in \Gamma_c(S|_U, p^*\mathcal{A})$ then $\iota_2^{-1} \circ \iota_1(\phi \cdot f)(u) = \iota_2^{-1}(\phi \cdot f)(u)$.[3] Now $\iota_2^{-1}(\phi \cdot f)(u)$ is just

[3]Recall that, because the quotient map is given by restriction on sections, $g(u)$ is the restriction of g to S_u.

the restriction of $\phi \cdot f$ to $C_c(S_u, A(u))$ and therefore

$$\iota_2^{-1} \circ \iota_1(\phi \cdot f)(u) = \phi(u)f(u) = \phi \cdot (\iota_2^{-1} \circ \iota_1(f))(u).$$

Thus $\iota_2^{-1} \circ \iota_1$ is $C_0(U)$-linear.

Next, it follows from Proposition 5.20 that the restriction map factors to an isomorphism $\bar{\rho}_1$ of $A \rtimes S/\operatorname{Ex}(U)$ onto $A(C) \rtimes S|_C$. We also know from Proposition 5.14 that the restriction map factors to an isomorphism $\bar{\rho}_2$ of $A \rtimes S/I_C$ onto $A \rtimes S(C)$. Since $I_C = \operatorname{Ex}(U)$ we may form the isomorphism $\bar{\rho}_2 \circ \bar{\rho}_1^{-1}$ of $A(C) \rtimes S|_C$ onto $A \rtimes S(C)$. Suppose $f \in \Gamma_c(S_C, p^*\mathcal{A})$ and $\phi \in C_0(C)$. Choose $a \in A \rtimes S$ so that $\rho_1(a) = f$. Applying Proposition 5.14 to $C_0(S^{(0)})$ it follows that the restriction map from $C_0(S^{(0)})$ to $C_0(C)$ is surjective. In particular we can extend ϕ to an element to of $C_0(S^{(0)})$. Clearly $\rho_1(\phi \cdot a) = \phi \cdot f$. Thus

$$\bar{\rho}_2(\bar{\rho}_1^{-1}(\phi \cdot f))(u) = \rho_2(\phi \cdot a)(u) = \phi(u)a(u) = \phi(u)\rho_2(a)(u) = \phi \cdot \bar{\rho}_2(\bar{\rho}_1^{-1}(f))(u)$$

and therefore the isomorphism $\bar{\rho}_2 \circ \bar{\rho}_1^{-1}$ is $C_0(C)$-linear. □

Corollary 5.25. *Suppose (A, S, α) is a separable dynamical system, S is a group bundle, and that U is an open subset of $S^{(0)}$. Then $(A \rtimes_\alpha S(U))^\wedge \cong (A(U) \rtimes_\alpha S|_U)^\wedge$. Furthermore, if we view both of these sets as the disjoint union $\coprod_{u \in U}(A(u) \rtimes S_u)^\wedge$ then the identification is given by the identity. Corresponding statements hold if C is a closed subset of $S^{(0)}$.*

Proof. Suppose U is an open subset of $S^{(0)}$ and let $C = S^{(0)} \setminus U$. Let $\sigma : (A \rtimes S)^\wedge \to S^{(0)}$ be the map arising from $A \rtimes S$ as a $C_0(S^{(0)})$-algebra. It follows from Corollary 5.15 that we can identify $(A \rtimes S(U))^\wedge$ with the set $\sigma^{-1}(U)$ and that, if we view both of these sets as $\coprod_{u \in U}(A(u) \rtimes S_u)^\wedge$ then the identification is given by the identity. Thus it suffices to show that we can identify $\sigma^{-1}(U)$ and $A(U) \rtimes S|_U$ in the appropriate manner. Now, the inclusion map ι extends to an isomorphism from $A(U) \rtimes S|_U$ to $\operatorname{Ex}(U)$ so that we can form the homeomorphism $\phi_1 : \operatorname{Ex}(U)^\wedge \to (A(U) \rtimes S_U)^\wedge$ given by $\phi_1(\pi) = \pi \circ \iota$. Furthermore, we can identify the set $P = \{\pi \in (A \rtimes S)^\wedge : \pi(\operatorname{Ex}(U)) \neq 0\}$ with $\operatorname{Ex}(U)^\wedge$ via the map $\phi_2 : P \to \operatorname{Ex}(U)^\wedge$ given by $\phi_2(\pi) = \pi|_{\operatorname{Ex}(U)}$. However $\operatorname{Ex}(U) = I_C$ and it was shown in the proof of Corollary 5.15 that in this case $P = \sigma^{-1}(U)$. Thus $\phi = \phi_1 \circ \phi_2$ is a homeomorphism from $\sigma^{-1}(U)$ onto $A(U) \rtimes S|_U$. Now suppose π is a representation of $A(u) \rtimes S_u$ and let π' be its lift to $A \rtimes S$ and π'' its lift to $A(U) \rtimes S|_U$. Furthermore, recall that the quotient map from $A \rtimes S$ to $A(u) \rtimes S_u$ is given by restriction on $\Gamma_c(S, p^*\mathcal{A})$. Then for $f \in \Gamma_c(S_U, p^*\mathcal{A})$, we have

$$\phi(\pi')(f) = \pi'(\iota(f)) = \pi(\iota(f)|_{S_u}) = \pi(f|_{S_u}) = \pi''(f)$$

Thus $\phi(\pi') = \pi''$ and if we identify $\sigma^{-1}(U)$ and $A(U) \rtimes S|_U$ with the disjoint union $\coprod_{u \in U}(A(u) \rtimes S_u)^\wedge$ then ϕ is given by the identity map.

Moving on, we can also use Corollary 5.15 to reduce to the problem of showing that $A(C) \rtimes S|_C$ can be identified with $\sigma^{-1}(C)$ in the appropriate fashion. Recall that the restriction map ρ factors to an isomorphism of $A \rtimes S/\operatorname{Ex}(U)$ with $A(C) \rtimes S|_C$ so that we can build a homeomorphism $\phi_1 : (A \rtimes S/\operatorname{Ex}(U))^\wedge \to (A(C) \rtimes S|_C)^\wedge$ by letting $\phi_1(\pi) = \pi \circ \overline{\rho}^{-1}$. Then we identify $Q = \{\pi \in (A \rtimes S)^\wedge : \operatorname{Ex}(U) \subset \ker \pi\}$ with $(A \rtimes S/\operatorname{Ex}(U))^\wedge$ via the map $\phi_2(\pi) = \bar{\pi}$ where $\bar{\pi}$ is the factorization of π to the quotient. Well $\operatorname{Ex}(U) = I_C$ and we showed in the proof of Corollary 5.15 that in this case $Q = \sigma^{-1}(C)$. Thus the desired homeomorphism is $\phi = \phi_1 \circ \phi_2$. Fix $u \in C$ and $\pi \in (A(u) \rtimes S_u)^\wedge$. Let π' be the lift of π to $A \rtimes S$ and π'' the lift to $A(C) \rtimes S_C$. Then given $f \in \Gamma_c(S, p^*\mathcal{A})$ we have

$$\phi(\pi')(\rho(f)) = \pi'(f) = \pi(f|_{S_u}) = \pi(\rho(f)|_{S_u}) = \pi''(\rho(f)).$$

Since ρ is surjective, this shows that $\phi(\pi') = \pi''$. Thus we can view ϕ as the identity map. $\square$

5.3 Unitary Actions

In this section we will discuss what it means for a groupoid to act trivially. The main goal will be to show that if the action is trivial then the crossed product reduces to a tensor product. As with group crossed products, trivial actions are going to be defined by unitaries.

Definition 5.26. Suppose S is a locally compact Hausdorff groupoid group bundle and A is a $C_0(S^{(0)})$-algebra. Then a *unitary action* of S on A is defined to be a collection $\{u_s\}_{s\in S}$ such that

- **(a)** $u_s \in UM(A(p(s)))$ for all $s \in S$,
- **(b)** $u_{st} = u_s u_t$ whenever $p(s) = p(t)$, and
- **(c)** $s \cdot a := u_s a$ defines a (strongly) continuous action of S on the associated upper-semicontinuous bundle $\mathcal{A}$.

The triple (A, S, u) is called a unitary dynamical system.

Remark 5.27. We show in Section 5.4 that if u is a unitary action of S on A then the restriction of u to S_v for $v \in S^{(0)}$ gives a unitary action of S_v on $A(v)$ in the sense of [Wil07, Definition 2.70]. Thus, Definition 5.26 is really just a "bundled" version of the notion of a unitary action of a group on a C^*-algebra.

As with groupoid dynamical systems there is an "unbundled" definition. However, we first take this opportunity to present some of the basic facts about multipliers of $C_0(X)$-algebras.

Lemma 5.28 ([Lee76, Lemma 2]). *Suppose that A is a $C_0(X)$-algebra and that $m \in M(A)$. Then for each $x \in X$ there exists a multiplier $m(x) \in M(A(x))$ such that $m(x)(a(x)) = m(a)(x)$. Conversely, if we are given $m_x \in M(A(x))$ for all $x \in X$ and if for each $a \in A$ there are elements $b, c \in A$ such that for all $x \in X$*

$$b(x) = m_x a(x) \quad \textit{and} \quad c(x) = m_x^* a(x), \tag{5.13}$$

then

- **(a)** *there is a $m \in M(A)$ such that $m(x) = m_x$ for all $x \in X$, and*
- **(b)** $\sup_{x \in X} \|m(x)\| = \|m\| < \infty$.

Remark 5.29. Condition (5.13) is equivalent to requiring that for all $a \in A$ there are elements $b, b' \in A$ such that $b(x) = m_x a(x)$ and $b'(x) = a(x) m_x$ for all $x \in X$.

Proof. Since any ideal I in A is also an ideal in $M(A)$ any multiplier m defines a multiplier m_I of A/I via $m_I(a + I) = m(a) + I$. The first portion of the lemma follows immediately. Now suppose we are given m_x as in the statement of the lemma and define a map $m : A \to A$ by $m(a)(x) = m_x a(x)$. The map m is well defined by assumption and it is easy to show that m is an adjointable A-linear operator on A with adjoint $m^*(a) = m_x^* a(x)$. Since we view multipliers as the adjointable A-linear operators on A_A we have established part (a).

Let $L = \sup_x \|m(x)\|$. Then, viewing A as $\Gamma_0(X, r^*\mathcal{A})$, we have

$$\begin{aligned} \|m(a)\| &= \sup_x \|m(a)(x)\| \\ &\leq \sup_x \|m(x)\| \|a(x)\| \\ &\leq \sup_x \|m(x)\| \|a\|. \end{aligned}$$

Thus $\|m\| \leq L$. Fix $\epsilon > 0$ and $x \in X$. We can find $b \in A(x)$ of norm one such that $\|m(x)b\| \geq \|m(x)\| - \epsilon$. Since the norm on $A(x)$ is the quotient norm there is an $a \in A$ with $a(x) = b$ and $\|a\| \leq 1 + \epsilon$. But then $\|m(a)\| \geq \|m(a)(x)\| \geq \|m(x)\| - \epsilon$ and it follows that

$$\|m\| \geq \frac{\|m(x)\| - \epsilon}{1 + \epsilon}$$

Since ϵ was arbitrary $\|m\| \geq \|m(x)\|$ for all $x \in X$ and $L \leq \|m\|$. □

Now we can present an equivalent form of Definition 5.26.

Proposition 5.30. *Suppose (A, S, u) is a unitary dynamical system. Then there is an element $u \in UM(p^*A)$ such that $u(s) = u_s$ for all $s \in S$. Conversely, if we*

have $u \in UM(p^*A)$ then there are elements $u_s \in UM(A(p(s)))$ for all $s \in S$ and if $u_{st} = u_s u_t$ whenever $p(s) = p(t)$ then $\{u_s\}$ defines a unitary action of S on A.

Proof. Suppose (A, G, u) is a unitary action and $f \in p^*A$. We need to show that

$$h(s) := u_s f(s), \qquad\qquad g(s) := u_s^* f(s)$$

define elements of p^*A. The continuity of h is obvious from condition (c) of Definition 5.26. Suppose $s_i \to s$ and $a_i \to a$. First, observe that condition (b) of Definition 5.26 guarantees that $u_{s^{-1}} = u_s^{-1} = u_s^*$ for all $s \in S$. Now, we know $s_i^{-1} \to s^{-1}$ and therefore

$$u_{s_i^{-1}} a_i = u_{s_i}^* a_i \to u_s^* a = u_{s^{-1}} a.$$

It follows immediately that g is continuous as well. Furthermore,

$$\|h(s)\| = \|u_s f(s)\| = \|f(s)\| = \|u_s^* f(s)\| = \|g(s)\|$$

so that both h and g must vanish at infinity because f does. Thus $h, g \in p^*A$. Hence Lemma 5.28 implies that there is a multiplier u such that $u(f)(s) = u_s f(s)$ for all $s \in S$. Since each u_s is a unitary, it is clear that u must be a unitary.

Next, suppose we are given $u \in UM(p^*A)$. Then, via Lemma 5.28, we know there exists multipliers u_s such that $u_s(f(s)) = u(f)(s)$. However, since u is a unitary each u_s must be as well. Furthermore, condition (b) of Definition 5.26 holds by assumption. All that is left is to show the action is continuous. Suppose $s_i \to s$ and $a_i \to a$ such that $p(s_i) = p(a_i)$ and $p(s) = p(a)$. Choose $f \in p^*A$ such that $f(s) = a$. Then $u(f) \in p^*A$ and $u(f)(s) = u_s(f(s)) = u_s a$. Furthermore, since $f(s_i) - a_i \to 0$, we have

$$\|u_{s_i} a_i - u(f)(s_i)\| = \|a_i - f(s_i)\| \to 0.$$

It follows from Proposition 3.10 that the action is continuous. □

Next, given a unitary dynamical system we can form an associated groupoid dynamical system as follows.

Remark 5.31. Suppose A is a C^*-algebra and $u \in UM(A)$. Then u defines an automorphism on A via conjugation. This automorphism is denoted $\operatorname{Ad} u$ and is given by $\operatorname{Ad} u(a) = uau^*$.

Proposition 5.32. *Suppose (A, S, u) is a unitary dynamical system. Then the collection $\{\operatorname{Ad} u_s\}_{s\in S}$ defines a groupoid action of S on A.*

Proof. Given a unitary action let u be the corresponding element of $UM(p^*A)$ guaranteed by Proposition 5.30. Then define $\operatorname{Ad} u : p^*A \to p^*A$ by $\operatorname{Ad} u(f) = ufu^*$. Clearly $\operatorname{Ad} u$ is a $C_0(S^{(0)})$-linear automorphism of $p^*(A)$. As in Proposition 3.50

there exists isomorphisms $(\operatorname{Ad} u)_s : A(p(s)) \to A(p(s))$ for all $s \in S$. Furthermore, these isomorphisms are given by

$$(\operatorname{Ad} u)_s(f(s)) = \operatorname{Ad}_u(f)(s) = ufu^*(s) = u_s f(s) u_s^*.$$

Thus $(\operatorname{Ad} u)_s = \operatorname{Ad} u_s$. Finally if $p(s) = p(t)$ then

$$\operatorname{Ad} u_s \circ \operatorname{Ad} u_t(a) = u_s u_t a u_t^* u_s^* = \operatorname{Ad} u_{st}(a).$$

It now follows from Proposition 3.50 that $\operatorname{Ad} u$ is an action of S on A. □

This allows us to define a special class of groupoid actions which we will eventually see are the aforementioned "trivial" dynamical systems.

Definition 5.33. Suppose S is a group bundle and A is a $C_0(S^{(0)})$-algebra. Then a dynamical system (A, S, α) is said to be *unitary* or *unitarily implemented* if there exists a unitary action u of S on A such that $\alpha = \operatorname{Ad} u$.

At this point we need to make a brief detour through the notion of equivalent actions. The following construction will play the role of isomorphism for dynamical systems.

Definition 5.34. Suppose G is a locally compact Hausdorff groupoid and A is a $C_0(X)$-algebra. Furthermore, suppose α and β are actions of G on A. Then we say that α and β are *exterior equivalent* if there is a collection $\{u_\gamma\}_{\gamma \in G}$ such that

(a) $u_\gamma \in UM(A(r(\gamma)))$ for all $\gamma \in G$,

(b) $u_{\gamma\eta} = u_\gamma \overline{\alpha}_\gamma(u_\eta)$ for all $\gamma, \eta \in G$ such that $s(\gamma) = r(\eta)$,[4]

(c) the map $(\gamma, a) \mapsto u_\gamma a$ is jointly continuous on the set $\{(\gamma, a) \in G \times \mathcal{A} : r(\gamma) = q(a)\}$, and

(d) $\beta_\gamma = \operatorname{Ad} u_\gamma \circ \alpha_\gamma$ for all $\gamma \in G$.

The following lemma expands on condition (b) above and gives us a formula for inversion.

Lemma 5.35. *Suppose that (A, G, α) and (A, G, β) are exterior equivalent dynamical systems and that the equivalence is implemented by $\{u_\gamma\}$. Then*

(a) $u_w = \mathrm{id}$ *for all $w \in G^{(0)}$, and*

(b) $u_{\gamma^{-1}} = \overline{\alpha}_\gamma^{-1}(u_\gamma^*)$ *for all $\gamma \in G$.*

[4] Here $\overline{\alpha}_\gamma$ denotes the canonical extension of α_γ to the multiplier algebra.

Proof. If $w \in G^{(0)}$ then $\alpha_w = \mathrm{id}$ so that

$$u_w = u_{w^2} = u_w \overline{\alpha}_w(u_w) = u_w^2.$$

Since u_w is a unitary this implies $u_w = \mathrm{id}$. Now, given $\gamma \in G$ we have

$$\mathrm{id} = u_{\gamma^{-1}\gamma} = u_{\gamma^{-1}} \overline{\alpha}_{\gamma^{-1}}(u_\gamma).$$

After recalling that $\alpha_{\gamma^{-1}} = \alpha_\gamma^{-1}$ and that $\overline{\alpha}_\gamma^{-1}(u_\gamma)$ is a unitary we conclude

$$\overline{\alpha}_\gamma^{-1}(u_\gamma^*) = \overline{\alpha}_\gamma^{-1}(u_\gamma)^* = u_{\gamma^{-1}}.$$ □

As before, we present an alternate definition which removes the bundle theory.

Proposition 5.36. *Suppose α and β are exterior equivalent actions of the locally compact groupoid G on the $C_0(G^{(0)})$-algebra A with the collection $\{u_\gamma\}$ implementing the equivalence. Then there is an element $u \in UM(r^*A)$ such that $u(f)(\gamma) = u_\gamma f(\gamma)$ for all $f \in r^*A$ and $\gamma \in G$.*

Conversely, if $u \in UM(r^*A)$ then there are $u_\gamma \in UM(A(r(\gamma)))$ for all $\gamma \in G$. If $u_{\gamma\eta} = u_\gamma \overline{\alpha}_\gamma(u_\eta)$ whenever $s(\gamma) = r(\eta)$ and $\beta_\gamma = \mathrm{Ad}\, u_\gamma \circ \alpha_\gamma$ for all $\gamma \in G$ then α and β are exterior equivalent.

Proof. Suppose α, β and $\{u_\gamma\}$ are as in the first part of the proposition and let $\mathcal{A}$ be the upper-semicontinuous bundle associated to A. Given $f \in r^*A$ we must show that

$$g(\gamma) := u_\gamma f(\gamma), \qquad\qquad h(\gamma) := u_\gamma^* f(\gamma)$$

define elements of r^*A. It is clear from condition (c) of Definition 5.34 that g defines a continuous section of $r^*\mathcal{A}$. Showing that h is continuous takes a little more work. Suppose $\gamma_i \to \gamma$ in G and $a_i \to a$ in $\mathcal{A}$ such that $r(\gamma_i) = q(a_i)$ for all i and $r(\gamma) = q(a)$. It follows that $\gamma_i^{-1} \to \gamma^{-1}$ and $a_i^* \to a^*$. Furthermore, since α is a continuous action, we have $\alpha_{\gamma_i}^{-1}(a_i^*) \to \alpha_\gamma^{-1}(a)$. It follows from the continuity of $\{u_\gamma\}$ that

$$u_{\gamma_i^{-1}} \alpha_{\gamma_i}^{-1}(a_i^*) \to u_{\gamma^{-1}} \alpha_\gamma^{-1}(a^*). \tag{5.14}$$

Applying Lemma 5.35 we conclude that

$$\alpha_{\gamma_i}^{-1}(u_{\gamma_i}^* a_i^*) \to \alpha_\gamma^{-1}(u_\gamma^* a^*). \tag{5.15}$$

If we apply the continuity of α with respect to $\gamma_i \to \gamma$ and (5.15) we obtain

$$u_{\gamma_i}^* a_i^* \to u_\gamma^* a^* \tag{5.16}$$

and therefore $a_i u_{\gamma_i} \to a u_\gamma$. It follows immediately that h is also a continuous section.

Furthermore, since u_γ is unitary for all $\gamma \in G$, we have

$$\|g(\gamma)\| = \|u_\gamma f(\gamma)\| = \|f(\gamma)\| = \|f(\gamma)u_\gamma\| = \|h(\gamma)\|.$$

Because f vanishes at infinity this is enough to show that $g, h \in r^*A$. Therefore we can conclude from Lemma 5.28 that there exists $u \in UM(r^*A)$ which has the required form.

Now suppose we are given $u \in UM(r^*A)$ with the properties listed in the second half of the proposition. It follows from Lemma 5.28 that there are $u_\gamma \in UM(A(r(\gamma)))$ for all $\gamma \in G$. Furthermore, by assumption, the only condition of Definition 5.34 which isn't satisfied is condition (c). Suppose $\gamma_i \to \gamma$ in G and $a_i \to a$ in $\mathcal{A}$ such that $r(\gamma_i) = q(a_i)$ and $r(\gamma) = q(a)$. Choose $f \in r^*A$ such that $f(\gamma) = a$. Then $u(f) \in r^*A$ and $u(f)(\gamma) = u_\gamma(f(\gamma)) = u_\gamma a$. Furthermore, since $f(\gamma_i) - a_i \to 0$, we have

$$\|u_{\gamma_i} a_i - u(f)(\gamma_i)\| = \|a_i - f(\gamma_i)\| \to 0.$$

The continuity condition now follows from Proposition 3.10. $\square$

The most important fact about exterior equivalent actions is the following

Proposition 5.37. *Suppose (A, G, α) and (A, G, β) are exterior equivalent separable groupoid dynamical systems with the equivalence implemented by $\{u_\gamma\}$. Then the map $\phi : \Gamma_c(G, r^*\mathcal{A}) \to \Gamma_c(G, r^*\mathcal{A})$ defined by*

$$\phi(f)(\gamma) = f(\gamma)u_\gamma^* \tag{5.17}$$

for all $\gamma \in G$ extends to an isomorphism from $A \rtimes_\alpha G$ onto $A \rtimes_\beta G$.

Proof. Let (A, G, α) and (A, G, β) be exterior equivalent dynamical systems with the equivalence implemented by $\{u_\gamma\}$. Use Proposition 5.36 to find $u \in UM(r^*A)$ such that $uf(\gamma) = u_\gamma f(\gamma)$ for all γ. Given $f \in \Gamma_c(G, r^*\mathcal{A})$ view f as an element of r^*A and define $\phi(f) = fu^*$. It is clear from the construction of u that ϕ is also given by (5.17). Furthermore $fu^* \in r^*A$ so that $\phi(f)$ is a continuous section. However, it follows from (5.17) that $\phi(f)$ is compactly supported as well.

Obviously, ϕ is linear. We would like to show that it is a $*$-homomorphism. Given $f, g \in \Gamma_c(G, r^*\mathcal{A})$ we have

$$\begin{aligned}
\phi(f) * \phi(g)(\gamma) &= \int_G f(\eta)u_\eta^* \beta_\eta(g(\eta^{-1}\gamma)u_{\eta^{-1}\gamma}^*) d\lambda^{r(\gamma)}(\eta) \\
&= \int_G f(\eta)u_\eta^* u_\eta \alpha_\eta(g(\eta^{-1}\gamma))\overline{\alpha}_\eta(u_{\eta^{-1}}^* \overline{\alpha}_\eta^{-1}(u_\gamma))^* u_\eta^* d\lambda^{r(\gamma)}(\eta) \\
&= \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))u_\gamma^* \overline{\alpha}_\eta(u_{\eta^{-1}}^*) u_\eta^* d\lambda^{r(\gamma)}(\eta).
\end{aligned}$$

Applying Lemma 5.35 to our calculation we obtain

$$\begin{aligned}\phi(f) * \phi(g)(\gamma) &= \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))u_\gamma^* u_\eta u_\eta^* d\lambda^{r(\gamma)}(\eta) \\ &= f * g(\gamma)u_\gamma^* = \phi(f * g)(\gamma).\end{aligned}$$

We can also use Lemma 5.35 to show that

$$\begin{aligned}\phi(f)^*(\gamma) &= \beta_\gamma(f(\gamma^{-1})u_{\gamma^{-1}}^*)^* = u_\gamma\alpha_\gamma(u_{\gamma^{-1}}f(\gamma^{-1})^*)u_\gamma^* \\ &= u_\gamma\overline{\alpha}_\gamma(u_{\gamma^{-1}})\alpha_\gamma(f(\gamma^{-1})^*)u_\gamma^* \\ &= u_\gamma u_\gamma^* f^*(\gamma)u_\gamma^* = \phi(f^*)(\gamma).\end{aligned}$$

Thus ϕ is a $*$-homomorphism.

Next, observe that given $f \in \Gamma_c(G, r^*\mathcal{A})$ we have

$$\|\phi(f)(\gamma)\| = \|f(\gamma)u_\gamma^*\| = \|f(\gamma)\|.$$

It follows quickly that ϕ is continuous with respect to the inductive limit topology and therefore Corollary 3.134 implies that ϕ extends to a $*$-homomorphism from $A \rtimes_\alpha G$ into $A \rtimes_\beta G$. We can define an inverse ψ for ϕ on $\Gamma_c(G, r^*\mathcal{A})$ by $\psi(f)(\gamma) = f(\gamma)u_\gamma$. An argument nearly identical to the above shows that ψ extends to a $*$-homomorphism on $A \rtimes_\beta G$. Since ϕ and ψ are inverses on a dense subset they are inverses on the entire algebra and ϕ is an isomorphism. □

Moving on, our statement that the unitary actions are "trivial" dynamical systems is supported by the next lemma. However, let us first introduce an action which is as trivial as possible.

Example 5.38. Suppose S is a locally compact group bundle and A is a $C_0(S^{(0)})$-algebra. Consider the identity map $\mathrm{id} : p^*A \to p^*A$. This isomorphism is clearly $C_0(S^{(0)})$-linear. Furthermore, $\mathrm{id}_s : A(p(s)) \to A(p(s))$ is the identity map for all $s \in S$. Therefore $\mathrm{id}_{st} = \mathrm{id}_s \circ \mathrm{id}_t$ and Proposition 3.50 implies that the collection of identity maps $\mathrm{id}_s : A(p(s)) \to A(p(s))$ defines an action of S on A. Observe that group bundles are the only groupoids which can act trivially in this way. If the source and range map are not equal then s^*A is not equal to r^*A and we cannot use the identity map to induce a groupoid action.

Lemma 5.39. *If (A, S, α) is a unitary dynamical system then it is exterior equivalent to the trivial system (A, S, id).*

Proof. Suppose α is implemented by the unitaries $\{u_s\}$. We claim that $\{u_s\}$ also implements an equivalence between id and α. Condition (a) of Definition 5.34 holds

by assumption, as does condition (c). Next observe that if $p(s) = p(t)$ then

$$u_{st} = u_s u_t = u_s \overline{\mathrm{id}}_s(u_t)$$

so that condition (b) holds as well. Finally we check that

$$\alpha_s(a) = \operatorname{Ad} u_s(a) = \operatorname{Ad} u_s \circ \mathrm{id}_s(a). \qquad \square$$

Remark 5.40. The curious reader may wonder why we have only defined unitary actions for a special class of groupoids. Unitary actions should always be equivalent to the trivial action and, as stated in Example 5.38, the trivial action only makes sense for group bundles. Thus, it only makes sense to define unitary actions for group bundles.

5.3.1 Tensor Products

We want to show that crossed products of unitary dynamical systems are tensor products. However, we are working with fibred objects so we need to use a tensor product which respects the bundle structure on the algebras. It is assumed that the reader is familiar with the basics of C^*-algebraic tensor products. In particular we will cite [RW98, Appendix B] frequently. In fact, just for reference, we reproduce the following

Proposition 5.41 ([RW98, Theorem B.27])**.** *Suppose A and B are C^*-algebras. Then there are nondegenerate homomorphisms $\iota_A : A \to M(A \otimes_{\max} B)$ and $\iota_B : B \to M(A \otimes_{\max} B)$ such that*

- **(a)** $\iota_A(a)\iota_B(b) = \iota_B(b)\iota_A(a) = a \otimes b$ *for $a \in A$ and $b \in B$,*
- **(b)** *if ϕ and ψ are representations of A and B with commuting ranges then there is a representation $\phi \otimes_{\max} \psi$ of $A \otimes_{\max} B$ such that*

  $$\phi \otimes_{\max} \psi(\iota_A(a)\iota_B(b)) = \phi(a)\psi(b)$$

  for $a \in A$ and $b \in B$,
- **(c)** $A \otimes_{\max} B = \overline{\operatorname{span}}\{\iota_A(a)\iota_B(b) : a \in A, b \in B\}$.

If D is a C^*-algebra and $j_A : A \to M(D)$ and $j_B : B \to M(D)$ are homomorphisms satisfying the analogues of (a),(b) and (c) then there is an isomorphism θ of $A \otimes_{\max} B$ onto D such that $\phi(a \otimes b) = j_A(a)j_B(b)$.

The proper notion of a “fibred” tensor product is the balanced tensor product, defined below.

Definition 5.42. Suppose A and B are $C_0(X)$-algebras and let $A \otimes_{\max} B$ denote the (maximal) tensor product of A and B. The *balancing ideal* I_X is the ideal in $A \otimes_{\max} B$ generated by

$$\{f \cdot a \otimes b - a \otimes f \cdot b : f \in C_0(X), a \in A, b \in B\}.$$

The *balanced tensor product* $A \otimes_{C_0(X)} B$ is defined to be the quotient $A \otimes_{\max} B/I_X$.

Remark 5.43. If we view two C^*-algebras A and B as being $C_0(\{\mathrm{pt}\})$-algebras then the balanced tensor product is just the usual maximal tensor product.

Remark 5.44. Our tensor products will generally be maximal tensor products [RW98, Appendix B]. However, most of the time we will be working with nuclear C^*-algebras so that we will not have to make this distinction.

Moving on, one of the key facts about the balanced tensor product is that, at least for nice C^*-algebras, its spectrum is the fibre product of the spectra of its components. This proposition is a reproduction of [RW85, Lemma 1.1].

Proposition 5.45. *Suppose A and B are separable $C_0(X)$-algebras and that either A or B is nuclear. Define the bundle product of $\widehat{A}$ with $\widehat{B}$ to be*

$$\widehat{A} \times_X \widehat{B} := \{(\pi, \rho) \in \widehat{A} \times \widehat{B} : \sigma_A(\pi) = \sigma_B(\rho)\}.$$

(a) *The map $(\pi, \rho) \mapsto \pi \otimes_\sigma \rho$ induces a homeomorphism Φ of $\widehat{A} \times_X \widehat{B}$ onto its range in $(A \otimes_{C_0(X)} B)^\wedge$.*

(b) *If either A or B is GCR then this homeomorphism is surjective.*

Proof. Since at least one of A or B is nuclear there is a unique tensor product $A \otimes B$. Furthermore, we cite [RW98, Theorem B.45] to see that the map $(\pi, \rho) \mapsto \pi \otimes_\sigma \rho$ induces a homeomorphism Φ of $\widehat{A} \times \widehat{B}$ onto its range in $(A \otimes B)^\wedge$ and is surjective if either A or B is GCR. Since $A \otimes_{C_0(X)} B$ is a quotient of $A \otimes B$ by the balancing ideal I, we can identify $(A \otimes_{C_0(X)} B)^\wedge$ with the closed set $\{R \in (A \otimes B)^\wedge : I \subset \ker R\}$. If $\pi \in \widehat{A}$ and $\rho \in \widehat{B}$ then $\pi \otimes_\sigma \rho(I) = 0$ if and only if $\pi(\phi \cdot a)\rho(b) = \pi(a)\rho(\phi \cdot b)$ for all $a \in A$, $b \in B$ and $\phi \in C_0(X)$. Let $x = \sigma_A(\pi)$ so that π factors to a representation $\overline{\pi}$ of $A(x)$ and $y = \sigma_B(\rho)$ so that ρ factors to a representation $\overline{\rho}$ of $B(y)$. Then $\pi(\phi \cdot a)\rho(b) = \pi(a)\rho(\phi \cdot b)$ if and only if

$$\phi(x)\overline{\pi}(a(x))\overline{\rho}(b(y)) = \phi(y)\overline{\pi}(a(x))\overline{\rho}(b(y)) \tag{5.18}$$

for all $\phi \in C_0(X)$, $a \in A$, and $b \in B$. However (5.18) holds if and only if $x = y$. Thus the restriction of Φ to the closed set $\widehat{A} \times_X \widehat{B}$ maps onto $(A \otimes_{C_0(X)} B)^\wedge$. □

Recall that in Definition 3.37 we defined the pull back of a C^*-algebra to be the section algebra of the pull back of the associated bundle. This is not how the pull

back is classically defined. However, we now have the tools to prove the following proposition, which brings us back to the usual definition.

Proposition 5.46 ([RW85, Proposition 1.3]). *Suppose X and Y are locally compact Hausdorff spaces, A is a $C_0(X)$-algebra and $\tau : Y \to X$ is a continuous surjection. Then the pull back algebra $\tau^* A$ is isomorphic to the balanced tensor product $C_0(Y) \otimes_{C_0(X)} A$.*

Proof. First, recall that if $\tau : Y \to X$ is a continuous surjection then we can view $C_0(Y)$ as a $C_0(X)$-algebra as in Example 3.17. Let $\iota : C_0(Y) \odot A \to \Gamma_0(Y, \tau^*\mathcal{A})$ be such that $\iota(f \otimes a)(y) = f(y)a(\tau(y))$.[5] It is clear that this defines a continuous section and $\iota(f \otimes a)$ vanishes at infinity because f does and a is bounded. It is straightforward to show that ι is a $*$-homomorphism and it follows quickly from Proposition 3.40 that ι maps onto a dense subset. Finally, because ι is a homomorphism, pulling back the uniform norm on $\Gamma_0(Y, \tau^*\mathcal{A})$ defines a C^*-seminorm on $C_0(Y) \odot A$ by $\|f \otimes a\|_\iota = \|\iota(f \otimes a)\|_\infty$. However this implies that $\|\iota(f \otimes a)\|_\infty \leq \|f \otimes a\|_{\max}$ and that ι extends to a representation of $C_0(Y) \otimes A$. Furthermore we clearly have

$$\iota(\phi \cdot f \otimes a)(y) = \phi(\tau(y))f(y)a(\tau(y)) = \iota(f \otimes \phi \cdot a)(y)$$

for all $y \in Y$. Hence ι vanishes on the balancing ideal and factors to a homomorphism on the balanced tensor product $C_0(Y) \otimes_{C_0(X)} A$, which we also denote by ι.

Now suppose R is an irreducible representation of $C_0(Y) \otimes_{C_0(X)} A$. Since $C_0(Y)$ is abelian it is both nuclear and GCR so that by Proposition 5.45 there exists $y \in Y$ and $\pi \in \widehat{A}$ such that $R = \mathrm{ev}_y \otimes \sigma\pi$ where ev_y is the evaluation representation. Furthermore we must have $\sigma(\pi) = \tau(y)$ so that π factors to a representation $\overline{\pi}$ of $A(\tau(y))$. If π acts on $\mathcal{H}$ then $\mathrm{ev}_y \otimes_\sigma \pi$ acts on $\mathbb{C} \otimes \mathcal{H}$, which we identify with $\mathcal{H}$, via $\mathrm{ev}_y \otimes_\sigma \pi(f \otimes a) = f(y)\pi(a)$. We may now compute for $\sum_i f_i \otimes a_i \in C_0(Y) \odot A$

$$\begin{aligned}
\left\| \mathrm{ev}_y \otimes_\sigma \pi \left(\sum_i f_i \otimes a_i \right) \right\| &= \left\| \sum_i f_i(y)\pi(a_i) \right\| = \left\| \pi \left(\sum_i f_i(y) a_i \right) \right\| \\
&= \left\| \overline{\pi} \left(\sum_i f_i(y) a_i(\tau(y)) \right) \right\| = \left\| \overline{\pi} \left(\sum_i f_i \otimes a_i(y) \right) \right\| \\
&\leq \left\| \sum_i f_i \otimes a_i(y) \right\| \leq \left\| \iota \left(\sum_i f_i \otimes a_i \right) \right\|_\infty .
\end{aligned}$$

Since this is true for every irreducible representation of $C_0(Y) \otimes_{C_0(X)} A$ we conclude $\|\sum_i f_i \otimes a_i\| \leq \|\iota(\sum_i f_i \otimes a_i)\|_\infty$. It follows that ι is isometric on a dense subset and

[5] In other words, view $f \otimes a$ as an elementary tensor in $\tau^* A$.

therefore must be isometric everywhere. Hence ι is an isomorphism of $C_0(Y) \otimes_{C_0(X)} A$ with $\tau^* A$. □

The last thing we want to show about balanced tensor products is that the resulting algebra is still a $C_0(X)$-algebra. This is, perhaps, unsurprising considering how balanced tensor products work for modules. Of course, it follows from Proposition 5.45 that, at least for nice algebras, the spectrum of the balanced tensor product is the bundle product of the spectra of its components. The $C_0(X)$-algebra structure then follows from Theorem 3.25. However, this construction can be done in greater generality.

Proposition 5.47. *Suppose A and B are $C_0(X)$-algebras. Then $A \otimes_{C_0(X)} B$ is a $C_0(X)$-algebra with the action characterized by*

$$\phi \cdot (a \otimes b) := (\phi \cdot a) \otimes b = a \otimes (\phi \cdot b)$$

Proof. Suppose Φ_A and Φ_B implement the $C_0(X)$-algebra structure on A and B respectively. Then Φ_A and Φ_B are nondegenerate homomorphisms into $ZM(A)$ and $ZM(B)$ respectively. Let ι_A and ι_B be the nondegenerate homomorphisms from Proposition 5.41 and let $\pi : A \otimes B \to A \otimes_{C_0(X)} B$ be the quotient map. Consider the map

$$\Psi = \overline{\pi} \circ \overline{\iota}_A \circ \Phi_A.$$

This is certainly a homomorphism and one can check, given $\phi \in C_0(X)$, $a \in A$ and $b \in B$, that

$$\begin{aligned} \Psi(\phi)(a \otimes b) &= \pi(\overline{\iota}_A(\Phi_A(\phi))\iota_A(a)\iota_B(b)) = \pi(\iota_A(\phi \cdot a)\iota_B(b)) \\ &= (\phi \cdot a) \otimes b = a \otimes (\phi \cdot b). \end{aligned} \tag{5.19}$$

Thus Ψ has the desired form. Now we need to see that it maps into the center of the multiplier algebra. Using Lemma 4.30 and linearity it will suffice to show that

$$\Psi(\phi)((a \otimes b)(c \otimes d)) = (a \otimes b)(\Psi(\phi)(c \otimes d))$$

for all $a, c \in A$ and $b, d \in B$. However, by (5.19) we have

$$\Psi(\phi)(ac \otimes bd) = (\phi \cdot ac) \otimes bd = (a \otimes b)((\phi \cdot c) \otimes d) = (a \otimes b)(\Psi(\phi)(c \otimes d)).$$

Thus Ψ maps into the center of the multiplier algebra. The last thing we need to show is that Ψ is nondegenerate. It will suffice to show that elements of the form $\phi \cdot a \otimes b$ span a dense subset. Since elements of the form $\phi \cdot a$ span a dense subset in A, we are done. □

Remark 5.48. We won't use this fact directly, and so do not prove it here, but it is clear enough that $(A \otimes_{C_0(X)} B)(x) = A(x) \otimes B(x)$ for all $x \in X$.

Let us get back to the matter at hand. We now are able to prove the main theorem concerning unitary actions, which is that they have trivial crossed products.

Theorem 5.49. *Suppose (A, S, α) is a separable unitary dynamical system with α implemented by u. Then there is a $C_0(S^{(0)})$-linear isomorphism $\phi : C^*(S) \otimes_{C_0(S^{(0)})} A \to A \rtimes_\alpha S$ which is characterized for $a \in A$ and $f \in C_c(S)$ by*

$$\phi(f \otimes a)(s) = f(s)a(p(s))u_s^* \tag{5.20}$$

Remark 5.50. Since ϕ is $C_0(S^{(0)})$-linear it factors, by Proposition 3.20, to isomorphisms $\phi_u : C^*(S_u) \otimes A(u) \to A(u) \rtimes S_u$. It is not difficult to check that these are the usual isomorphisms that arise from unitary actions [Wil07, Lemma 2.73].

Proof. First, let β be a Haar system for S and consider the trivial action id of S on A. Given $a \in A$ and $f \in C_c(S)$ define $\iota(f \otimes a)(s) := f(s)a(p(s))$. In other words, view $f \otimes a$ as an "elementary tensor" in $\Gamma_c(S, p^*\mathcal{A})$ in the sense of Proposition 3.38. It follows that $\iota(f \otimes a) \in \Gamma_c(S, p^*\mathcal{A})$. Extend ι to the algebraic tensor product $C_c(S) \odot A$ by linearity so that $\iota : C_c(S) \odot A \to \Gamma_c(G, r^*\mathcal{A})$. Observe that $\operatorname{ran} \iota$ is dense with respect to the inductive limit topology. We would like to show that ι is a $*$-homomorphism. It is enough to do the calculations on elementary tensors. For $f, g \in C_c(S)$ and $a, b \in A$ we have

$$\begin{aligned} \iota(f \otimes a) * \iota(g \otimes b)(s) &= \int_S f(t)a(p(t))g(t^{-1}s)b(p(t))d\beta^{p(s)}(t) \\ &= \int_S f(t)g(t^{-1}s)d\beta^{p(s)}(t)ab(p(s)) \\ &= \iota(f * g \otimes ab)(s). \end{aligned}$$

and

$$\iota(f \otimes a)^*(s) = (f(s^{-1})a(p(s)))^* = \overline{f(s^{-1})}a^*(p(s)) = \iota(f^* \otimes a^*)(s).$$

Thus ι is a $*$-homomorphism.

Now we check that ι is bounded. Suppose $(S^{(0)} * \mathfrak{H}, \mu, \pi, U)$ is a covariant representation of (A, S, id). Then U is a groupoid representation of S and we can form the integrated representation as in Proposition 4.18 which we also denote by U. Let $\pi = \int_{S^{(0)}}^{\oplus} \pi_u d\mu(u)$ be a decomposition of π. Since (π, U) is covariant we must have, for all $a \in A$ and almost every $s \in S$,

$$\pi_{p(s)}(a(p(s)))U_s = U_s\pi_{p(s)}(a(p(s))). \tag{5.21}$$

However, we can now compute for $f \in C_c(G)$ and $h \in \mathcal{L}^2(S^{(0)} * \mathfrak{H}, \mu)$ that

$$\begin{aligned}(\pi(a)U(f))h(u) &= \pi_u(a(u))U(f)h(u) \\ &= \int_S \pi_u(a(u))f(s)U(s)h(u)\Delta(s)^{-\frac{1}{2}}d\beta^u(s) \\ &= \int_S f(s)U(s)\pi(a(u))h(u)\Delta(s)^{-\frac{1}{2}}d\beta^u(s) \\ &= (U(f)\pi(a))h(u).\end{aligned}$$

We can extend this by continuity to all $f \in C^*(S)$ and conclude that π and U are commuting representations of A and $C^*(S)$. It follows from Proposition 5.45 that there exists a representation $U \otimes \pi$ on $C^*(S) \otimes A$ such that

$$U \otimes \pi(f \otimes a) = U(f)\pi(a).$$

Given $f \in C_c(G)$ and $a \in A$ we check that

$$\begin{aligned}\pi \rtimes U(\iota(f \otimes a))h(u) &= \int_S \pi_u(f(s)a(u))U_s h(u)\Delta(s)^{-\frac{1}{2}}d\beta^u(s) \qquad (5.22)\\ &= \pi_u(a(u))\int_S f(s)U_s h(u)\Delta(s)^{-\frac{1}{2}}d\beta^u(s) \\ &= \pi(a)U(f)h(u) = U \otimes \pi(f \otimes a)h(u).\end{aligned}$$

Using linearity, we conclude that $\pi \rtimes U(\iota(\xi)) = U \otimes \pi(\xi)$ for all $\xi \in C_c(S) \odot A$. Thus, given $\xi \in C_c(S) \odot A$,

$$\|\pi \rtimes U(\iota(\xi))\| = \|U \otimes \pi(\xi)\| \leq \|\xi\|.$$

Since this is true for all covariant representations (π, U), it follows that ι is bounded and extends to a homomorphism on $C^*(S) \otimes A$. Furthermore, since the range of ι is dense, it must be surjective. What's more, given $\phi \in C_0(S^{(0)})$, $f \in C_c(G)$ and $a \in A$ we have

$$\iota(\phi \cdot f \otimes a)(s) = \phi(p(s))f(s)a(p(s)) = \iota(f \otimes \phi \cdot a)(s).$$

It follows by continuity and linearity that ι factors through the balancing ideal and induces a homomorphism $\hat{\iota} : C^*(S) \otimes_{C_0(X)} A \to A \rtimes S$.

We would like to show that $\hat{\iota}$ is isometric. Suppose R is a faithful representation of $C^*(S) \otimes_{C_0(X)} A$ and let $\overline{R}$ be its lift to $C^*(S) \otimes A$. It follows [RW98, Corollary B.22] that there are commuting representations π and U of A and $C^*(S)$ such that $\overline{R} = U \otimes \pi$. Furthermore, since $U \otimes \pi$ contains the balancing ideal, a quick computation shows that $U(\phi \cdot f)\pi(a) = U(f)\pi(\phi \cdot a)$ for all $\phi \in C_0(S^{(0)})$, $f \in C^*(S)$, and $a \in A$.

Now, without loss of generality, we can use Theorem 4.19 to assume that U is the integrated form of some groupoid representation $(S^{(0)} * \mathfrak{H}, \mu, U)$. Furthermore we have for all $\phi \in C_0(S^{(0)})$, $a \in A$, and $f \in C^*(s)$

$$\begin{aligned}\pi(\phi \cdot a)U(f)h(u) &= U(\phi \cdot f)\pi(a)h(u) \\ &= \int_S \phi(u)f(s)U_s\pi(a)h(u)\Delta(s)^{-\frac{1}{2}}d\beta^u(s) \\ &= \phi(u)U(f)\pi(a)h(u) \\ &= T_\phi\pi(a)U(f)h(u)\end{aligned}$$

where T_ϕ is the diagonal operator associated to ϕ. Since U is nondegenerate this implies that π is $C_0(X)$-linear. Let $\pi = \int_{S^{(0)}}^{\oplus} \pi_u d\mu(u)$ be the decomposition of π and let ν be the measure on S induced by μ. All we need to do to prove that (π, U) is a covariant representation of (A, S, id) is to verify the covariance relation. In other words we need to show that (5.21) holds ν-almost everywhere. Let $\{a_i\}$ be a countable dense subset in A and e_l a special orthogonal fundamental sequence for $S^{(0)} * \mathfrak{H}$. Since the ranges of π and U commute, we have for all i, l, k and $f \in C_c(G)$

$$\begin{aligned}0 =&(\pi(a_i)U(f)e_l, e_k) - (U(f)\pi(a_i)e_l, e_k) \\ =& \int_S (f(s)\pi_{p(s)}(a_i(p(s)))U_s e_l(p(s)), e_k(p(s)))\Delta(s)^{-\frac{1}{2}}d\nu(s) \\ &- \int_S (f(s)U_s\pi_{p(s)}(a_i(p(s)))e_l(p(s)), e_k(p(s)))\Delta(s)^{-\frac{1}{2}}d\nu(s) \\ =& \int_S f(s)((\pi_{p(s)}(a_i(p(s)))U_s - U_s\pi_{p(s)}(a_i(p(s))))e_l(p(s)), e_k(p(s)))d\nu(s).\end{aligned}$$

This holds for all $f \in C_c(G)$ so that we may conclude for each i, l and k there exists a ν-null set $N_{i,l,k}$ such that

$$((\pi_{p(s)}(a_i(p(s)))U_s - U_s\pi_{p(s)}(a_i(p(s))))e_l(p(s)), e_k(p(s))) = 0 \tag{5.23}$$

for all $s \notin N_{i,l,k}$. However if we let $N = \bigcup_{i,l,k} N_{i,l,k}$ then N is still a ν-null set and for each $s \notin N$ (5.23) holds for all i, l and k. Since $\{e_l(p(s))\}$ is a basis (plus zero vectors) for each $p(s)$ this implies that for $s \notin N$ we have

$$\pi_{p(s)}(a_i(p(s)))U_s = U_s\pi_{p(s)}(a_i(p(s)))$$

for all i. It follows from the fact that $\{a_i\}$ is dense in A that this holds for all $a \in A$. Thus (π, U) is covariant representation of (A, G, id). Furthermore, we can reuse the computation in (5.22) to show that in this case $\pi \rtimes U \circ \iota = \pi \otimes U$. Given $\xi \in C^*(S) \otimes A$

let ξ' be its image in $C^*(S) \otimes_{C_0(X)} A$. We then have

$$\|\xi'\| = \|R(\xi')\| = \|U \otimes \pi(\xi)\| = \|\pi \rtimes U(\iota(\xi))\| \le \|\iota(\xi)\| = \|\hat{\iota}(\xi')\|.$$

It follows that $\hat{\iota}$ is isometric and is therefore an isomorphism.

To finish the proof, observe that because of Proposition 5.37 and Lemma 5.39, the map $\psi : A \rtimes_{\mathrm{id}} S \to A \rtimes_\alpha S$ given by $\psi(f)(s) = f(s)u_s^*$ is an isomorphism. Thus $\phi = \psi \circ \hat{\iota}$ is an isomorphism from $C^*(G) \otimes_{C_0(X)} A$ onto $A \rtimes_\alpha S$ and given $f \in C_c(S)$ and $a \in A$ we have

$$\phi(f \otimes a)(s) = \iota(f \otimes a)(s)u_s^* = f(s)a(p(s))u_s^*.$$

It follows quickly that ϕ is a $C_0(S^{(0)})$-linear isomorphism and we are done. $\square$

5.4 Locally Unitary Actions

Now that we have developed the theory of unitary actions we can modify Definition 5.26 and introduce some new concepts. The basic idea is that we weaken the continuity condition and see what kind of structure we have left. This material is a generalization of [PR84].

Definition 5.51. Suppose S is a group bundle and A is a $C_0(S^{(0)})$-algebra. Then a dynamical system (A, G, α) is said to be *pointwise unitary* if $\alpha|_{S_u}$ is unitarily implemented for each $u \in S^{(0)}$.

Notice that in a pointwise unitary dynamical system α_s is still given by conjugation by a unitary for all $s \in S$. What we have done is restrict the continuity of the unitaries to just the fibres. Of course, this should bring us back to the usual notion of a unitary group action, which we show in the following proposition.

Remark 5.52. Given a C^*-algebra A a function $f : X \to M(A)$ is said to be strictly continuous if $x \mapsto f(x)a$ is continuous for all $a \in A$.

Proposition 5.53. *Suppose G is a locally compact group and A is a C^*-algebra. Then a map $u : G \to UM(A)$ is strictly continuous if and only if the function $(s, a) \mapsto u_s a$ is continuous on $G \times A$. In particular, a unitary action of G on A is given by a strictly continuous homomorphism $u : G \to UM(A)$.*

Proof. The reverse direction is clear since strict continuity is weaker than joint continuity. Now suppose u is strictly continuous, $s_i \to s$, and $a_i \to a$. Then

$$\|u_{s_i}a_i - u_s a\| \le \|u_{s_i}(a_i - a)\| + \|u_{s_i}a - u_s a\| \le \|a_i - a\| + \|u_{s_i}a - u_s a\| \to 0$$

It follows that $u_{s_i}a_i \to u_s a$. The rest of the proposition follows. $\square$

The problem with pointwise unitary actions is that the unitaries tell you very little about the total space of the bundle structure. It will turn out to be more interesting if we pick a point "between" unitary and pointwise unitary.

Definition 5.54. Suppose S is a group bundle and A is a $C_0(S^{(0)})$-algebra. A dynamical system (A, S, α) is said to be *locally unitary* if there is an open cover $\{U_i\}_{i\in I}$ of $S^{(0)}$ such that $(A(U_i), S|_{U_i}, \alpha|_{S_{U_i}})$ is unitarily implemented for all $i \in I$.

Note that if S is a group bundle then every set in $S^{(0)}$ is S-invariant so the above definition makes sense. Our goal will be to analyze the exterior equivalence classes of abelian locally unitary actions on C^*-algebras with Hausdorff spectrum. In particular, the rest of the C^*-algebras in this section will have Hausdorff spectrum and we will view them as $C_0(\widehat{A})$-algebras as in Example 3.24. This will allow us to identify the spectrum of the crossed product for unitary actions.

Proposition 5.55. *Suppose S is an abelian, second countable, locally compact Hausdorff continuously varying group bundle, that A is a C^*-algebra with Hausdorff spectrum $S^{(0)}$ and that (A, S, α) is a unitary dynamical system. Let $\{u_s\}$ be the unitaries implementing α and for all $v \in S^{(0)}$ let π_v be the unique (up to equivalence) irreducible representation of $A(v)$. Define, for $\omega \in \widehat{S}$,*

$$\omega\overline{\pi}_{\hat{p}(\omega)}(u)(s) := \omega(s)\overline{\pi}_{\hat{p}(\omega)}(u_s).$$

Then the map $\phi : \widehat{S} \to (A \rtimes_\alpha S)^\wedge$ given by $\phi(\omega) = \pi_{\hat{p}(\omega)} \rtimes \omega\overline{\pi}_{\hat{p}(\omega)}(u)$ is a bundle homeomorphism.

Remark 5.56. Since we can't use u to denote both unitaries and units we will temporarily use x to denote elements of $S^{(0)}$.

Proof. Let (A, S, α) and u be as above. It follows from Theorem 5.49 that the map $\psi : C^*(S) \otimes_{C_0(S^{(0)})} A \to A \rtimes S$ characterized by $\psi(a \otimes f)(s) = f(s)au_s^*$ is an isomorphism. Hence, there is a homeomorphism $\phi_1 : (C^*(S) \otimes A)^\wedge \to (A \rtimes_{C_0(S^{(0)})} S)^\wedge$ such that $\phi_1(R) = R \circ \psi^{-1}$. Next, recall that we identify the dual group bundle $\widehat{S}$ with $C^*(S)^\wedge$. Since $C^*(S)$ is an abelian C^*-algebra, and is therefore GCR and nuclear, it follows from Proposition 5.45 that $\phi_2 : \widehat{S} \times_{S^{(0)}} \widehat{A} \to (C^*(S) \otimes_{C_0(S^{(0)})} A)^\wedge$ given by $\phi_2(\omega, \pi) = \omega \otimes_\sigma \pi$ is a homeomorphism. Recall that if π is a representation on $\mathcal{H}$ then $\omega \otimes_\sigma \pi$ is a representation on $\mathbb{C} \otimes \mathcal{H}$, which we will of course identify with $\mathcal{H}$, characterized by $\omega \otimes_\sigma \pi(f \otimes a) = \omega(f)\pi(a)$. Moving on, since $\widehat{A} = S^{(0)}$ we can define another homeomorphism $\phi_3 : \widehat{S} \to \widehat{S} \times_{S^{(0)}} \widehat{A}$ by $\phi_3(\omega) = (\omega, \pi_{\hat{p}(\omega)})$. Let $\phi = \phi_1 \circ \phi_2 \circ \phi_3$ and observe that $\phi : \widehat{S} \to (A \rtimes S)^\wedge$ is a homeomorphism. Furthermore given $\omega \in \widehat{S}$ we have $\phi(\omega) = \omega \otimes_\sigma \pi_{\hat{p}(\omega)} \circ \psi^{-1}$.

Now, fix $x \in S^{(0)}$ and $\omega \in \widehat{S}_x$ and define the map $U : S_x \to U(\mathcal{H})$ by $U_s = \omega(s)\overline{\pi}_x(u_s)$. Since u is a continuous action, and since ω is continuous, it follows quickly that U is a unitary representation of S_x. Furthermore we can compute for $a \in A(x)$ and $s \in S_x$ that

$$U_s\pi_x(a) = \omega(s)\pi_x(u_s a) = \omega(s)\pi_x(u_s a u_s^* u_s) = \pi_x(\alpha_s(a))U_s.$$

Thus (π_x, U) is a covariant representation of $(A(x), S_x, \alpha)$.[6] As such we can form the integrated representation $\pi_x \rtimes U$. Recall that $A \rtimes S$ is a $C_0(S^{(0)})$-algebra and that the restriction map ρ factors to an isomorphism between $A \rtimes S(x)$ and $A(x) \rtimes S_x$. Using the restriction map to view $\pi_x \rtimes U$ as a representation of $A \rtimes S$ we claim that $\pi_x \rtimes U = \phi(\omega)$. It will suffice to show that given an elementary tensor $f \otimes a$ then $\pi_x \rtimes U(\psi(f \otimes a)) = \omega \otimes_\sigma \pi_x(f \otimes a)$. We compute, observing that the modular function is one since S is abelian,

$$\begin{aligned}
\pi_x \rtimes U(\psi(f \otimes a))h &= \int_S \pi_x(f(s)a(x)u_s^*)\omega(s)\overline{\pi}_x(u_s)h d\beta^x(s) \\
&= \int_S f(s)\omega(s)d\beta^x(s)\pi_x(a(x))h \\
&= \omega(f)\pi_x(a)h = (\omega \otimes_\sigma \pi_x)(f \otimes a)h.
\end{aligned}$$

Thus $\phi(\omega) = \pi_x \rtimes U$ and since U is just an abbreviated notation for $\omega\overline{\pi}_x(u)$ we are almost done. All that is left is to show that ϕ preserves the fibres, but if ω is a representation of S_x then $\phi(\omega) = \pi_x \rtimes U$ is clearly a representation of the fibre $A(x) \rtimes S_x$. $\square$

Let us see what this implies in the weaker pointwise unitary case.

Corollary 5.57. *Suppose S is an abelian, second countable, locally compact Hausdorff continuously varying group bundle, that A is a C^*-algebra with Hausdorff spectrum $S^{(0)}$ and that (A, S, α) is a pointwise unitary dynamical system. Then we can view $(A \rtimes_\alpha S)^\wedge$ as a topological bundle over $S^{(0)}$ and for all $x \in S^{(0)}$ the fibre over x is isomorphic to $\widehat{S}_x$*

Proof. Recall $A \rtimes S$ is a $C_0(S^{(0)})$-algebra with fibres $A(x) \rtimes S_x$. By definition, α is unitarily implemented on S_x for all x. Thus we can apply Proposition 5.55 to each fibre and conclude that $(A(x) \rtimes S_x)^\wedge$ is homeomorphic to $\widehat{S}_x$. If we view $(A \rtimes S)^\wedge$ as a topological bundle with fibres $(A(x) \rtimes S_x)^\wedge$ then the result follows. $\square$

[6] We described what covariant representations looked like for group dynamical systems in Remark 4.3.

So if α is pointwise unitary then $(A \rtimes S)^\wedge$ is fibrewise isomorphic to the dual bundle $\widehat{S}$. A good question is to ask when is $(A \rtimes S)^\wedge$ a principal $\widehat{S}$-bundle in the sense of Section 2.1. Of course, the answer is hidden in the title of this section.

Theorem 5.58. *Suppose S is an abelian, second countable, locally compact Hausdorff, continuously varying group bundle, that A is a C^*-algebra with Hausdorff spectrum $S^{(0)}$ and that (A, S, α) is a locally unitary dynamical system. Let u^i implement α on $S|_{U_i}$ where $\{U_i\}$ is an open cover of $S^{(0)}$ and let $q : (A \rtimes_\alpha S)^\wedge \to S^{(0)}$ be the bundle map. Then for each i the map $\psi_i : \hat{p}_i^{-1}(U_i) \to q^{-1}(U_i)$ such that*

$$\psi_i(\omega) = \pi_{\hat{p}(\omega)} \rtimes \omega\overline{\pi}_{\hat{p}(\omega)}(u^i) \tag{5.24}$$

is a homeomorphism and the map γ_{ij} such that

$$\gamma_{ij}(p(s))(s) = \overline{\pi}_{p(s)}((u_s^i)^* u_s^j) \tag{5.25}$$

defines a continuous section of $\widehat{S}$. Furthermore these maps make $(A \rtimes S)^\wedge$ into a principal $\widehat{S}$-bundle with trivialization $(\mathcal{U}, \psi^{-1}, \gamma)$.

Proof. Let (A, G, α) be as in the statement of the theorem. Let $\{u^i\}$ implement α on $S|_{U_i}$ where U_i is an element of some open cover $\mathcal{U}$. Given an open set $U \in \mathcal{U}$ we identify each of $(A(U) \rtimes S|_U)^\wedge$, $(A \rtimes S(U))^\wedge$ and $q^{-1}(U)$ with the disjoint union $\coprod_{x\in U}(A(x) \rtimes S_x)^\wedge$. Furthermore Corollary 5.15 and Corollary 5.25 imply that this identification respects the topologies on all three spaces. In a similar fashion we identify each of $(C^*(S)(U))^\wedge$, $C^*(S|_U)^\wedge$ and $\hat{p}^{-1}(U)$ with the disjoint union $\coprod_{x\in U} \widehat{S}_x$ and again observe that this identification preserves the topologies.

Now, fix $U_i \in \mathcal{U}$. By assumption $\alpha|_{S|_{U_i}}$, denoted α whenever possible, is unitarily implemented by $\{u^i\}$ and as such Proposition 5.55 implies that the map $\psi_i : (S|_{U_i})^\wedge \to (A(U_i) \rtimes S|_{U_i})^\wedge$ defined via (5.24) is a homeomorphism. However, under the identifications made in the previous paragraph, we can view ψ_i as a map from $\hat{p}^{-1}(U_i)$ onto $q^{-1}(U_i)$. Furthermore, $q \circ \psi_i = \hat{p}$ since ψ_i is a bundle isomorphism. We define the trivializing maps on $(A \rtimes S)^\wedge$ to be $\phi_i = \psi_i^{-1}$. What's more, since $(A \rtimes S)^\wedge$ is locally homeomorphic to a locally compact Hausdorff space, we can conclude that $(A \rtimes S)^\wedge$ is locally compact Hausdorff.

Next, suppose $U_i, U_j \in \mathcal{U}$ and for each $x \in U_{ij}$ let π_x be the (unique) irreducible representation of $A(x)$. On $A(x) \rtimes S_x$ both u^i and u^j implement α so that we compute, for $s \in S_x$ and $a \in A(x)$,

$$\overline{\pi}_x((u_s^i)^* u_s^j)\pi_x(a) = \pi_x((u_s^i)^* u_s^j a) = \pi_x(\alpha_s^{-1}(\alpha_s(a))(u_s^i)^* u_s^j) = \pi_x(a)\overline{\pi}_x((u_s^i)^* u_s^j).$$

Since π_x is irreducible it follows [RW98, Lemma A.1] that $\gamma_{ij}(x)(s) := \overline{\pi}_x((u_s^i)^* u_s^j)$ is a scalar. Since u_s^i and u_s^j are unitaries $\gamma_{ij}(x)(s)$ must be a unitary as well and therefore

has modulus one. Next, observe that

$$\begin{aligned}\gamma_{ij}(x)(st) &= \overline{\pi}_x((u^i_{st})^* u^j_{st}) = \overline{\pi}_x((u^i_t)^*)\overline{\pi}_x((u^i_s)^* u^j_s)\overline{\pi}_x(u^j_t)\\ &= \overline{\pi}_x((u^i_t)^*)\gamma_{ij}(x)(s)\overline{\pi}_x(u^j_t) = \gamma_{ij}(x)(s)\gamma_{ij}(x)(t)\end{aligned}$$

Thus $\gamma_{ij}(x)$ is a homomorphism on S_x. Finally if $s_l \to s$ then, given $a \in A(x)$ and $h \in H$, we use the continuity of u^i and u^j to conclude

$$\gamma_{ij}(x)(s_l)\pi_x(a)h = \pi_x((u^i_{s_l})^* u^j_{s_l} a)h \to \pi_x((u^i_s)^* u^j_s a)h = \gamma_{ij}(x)(s)\pi_x(a)h.$$

Of course, this implies that $\gamma_{ij}(x)$ is continuous so that $\gamma_{ij}(x)$ is a character on S_x and the map γ_{ij} is a section of $\widehat{S}$ on U_{ij}. Next, we compute for $\omega \in \hat{p}^{-1}(U_{ij})$

$$\phi_i \circ \phi_j^{-1}(\omega) = \psi_i^{-1} \circ \psi_j(\omega) = \psi_i(\pi_{\hat{p}(\omega)} \rtimes \omega\overline{\pi}_{\hat{p}(\omega)}(u^j)). \tag{5.26}$$

Given $s \in S_{\hat{p}(\omega)}$ we have

$$\overline{\pi}_{\hat{p}(\omega)}(u^j_s) = \overline{\pi}_{\hat{p}(\omega)}(u^i_s)\overline{\pi}_{\hat{p}(\omega)}((u^i_s)^* u^j_s) = \gamma_{ij}(\hat{p}(\omega))(s)\overline{\pi}_{\hat{p}(\omega)}(u^i_s).$$

Applying this to (5.26) we obtain

$$\phi_i \circ \phi_j^{-1}(\omega) = \psi_i^{-1}(\pi_{\hat{p}(\omega)} \rtimes (\omega\gamma_{ij}(\hat{p}(\omega))\overline{\pi}_{\hat{p}(\omega)}(u^i))) = \omega\gamma_{ij}(\hat{p}(\omega)) \tag{5.27}$$

Thus (5.27) shows that the γ_{ij} are transition functions for the ϕ_i. Furthermore, suppose $u_l \to u$. Then

$$\gamma_{ij}(u_l) = \phi_i \circ \phi_j^{-1}(u_l) \to \phi_i \circ \phi_j(u) = \gamma_{ij}(u).$$

This suffices to show that γ_{ij} is continuous. It now follows that the trivialization $(\mathcal{U}, \phi, \gamma)$ makes $(A \rtimes S)^\wedge$ into a principal $\widehat{S}$-bundle. □

Of course this is little more than a curiosity unless we can use the principal bundle structure to tell us something about the action α. Fortunately, we can do just that.

Theorem 5.59. *Suppose S is an abelian, second countable, locally compact Hausdorff, continuously varying group bundle and that A has Hausdorff spectrum $S^{(0)}$. Two locally unitary actions (A, S, α) and (A, S, β) are exterior equivalent if and only if $(A \rtimes_\alpha S)^\wedge$ and $(A \rtimes_\beta S)^\wedge$ are isomorphic as $\widehat{S}$-bundles.*

Proof. Suppose α and β are equivalent locally unitary actions and the equivalence is implemented by the collection $\{u_s\}$. It follows from Proposition 5.37 that the map $\phi : A \rtimes_\alpha S \to A \rtimes_\beta S$ defined for $f \in \Gamma_c(S, p^*\mathcal{A})$ by $\phi(f)(s) = f(s)u_s^*$ is an isomorphism. As such it induces a homeomorphism $\Phi : (A \rtimes_\beta S)^\wedge \to (A \rtimes_\alpha S)^\wedge$

via the map $\Phi(\pi) = \pi \circ \phi$. Furthermore, ϕ is $C_0(S^{(0)})$-linear so that ϕ factors to an isomorphism on each of the fibres. This implies that, if I_x^α is the ideal such that $(A \rtimes_\alpha S)/I_x^\alpha = A \rtimes_\alpha S(x)$ and I_x^β is the corresponding ideal for β, then we must have $\phi(I_x^\alpha) = I_x^\beta$. Thus if $I_x^\beta \subset \ker \pi$ then $I_x^\alpha \subset \ker \pi \circ \phi$, and if q_α is the bundle map on $(A \rtimes_\alpha S)^\wedge$ and q_β is the corresponding map for β then we must have, by definition, $q_\beta(\pi) = q_\alpha(\Phi(\pi))$. Therefore Φ is a bundle isomorphism.

Next, let us establish some notation. Since α and β are both locally trivial we may as well pass to some common refinement and assume that there exists an open cover $\mathcal{U}$ of $S^{(0)}$ such that on $S|_{U_i}$ the unitary actions v^i and w^i implement α and β, respectively. Let ϕ_i and ψ_i be the trivializing maps induced by v^i and w^i, respectively. Furthermore given $x \in S^{(0)}$ let π_x be the (unique) irreducible representation of $A(x)$ associated to x. Now fix $U_i \in \mathcal{U}$ and $x \in U_i$. In order to conserve notation we will drop the i's on the v^i and w^i unless they are needed. Recall that $\beta_s = \operatorname{Ad} u_s \circ \alpha_s$ so that we can compute for $s \in S_x$

$$\begin{aligned}
u_s^* w_s v_s^* a &= u_s^* \beta_s(\alpha_s^{-1}(a)) w_s v_s^* \\
&= \operatorname{Ad}(u_s^*) \circ \beta_s \circ \alpha_s^{-1}(a) u_s^* w_s v_s^* \\
&= \operatorname{Ad}(u_s^*) \circ \operatorname{Ad}(u_s) \circ \alpha_s \circ \alpha_s^{-1}(a) u_s^* w_s v_s^* \\
&= a u_s^* w_s v_s^*.
\end{aligned}$$

It follows that $\beta_i(x)(s) := \overline{\pi}_x(u_s^* w_s v_s^*)$ commutes with $\pi_x(A(x))$. Since π_x is irreducible this implies that $\beta_i(x)(s)$ must be a scalar. Since u_s, w_s and v_s are all unitaries $\beta_i(x)(s)$ must have modulus one. Furthermore, it is straightforward to show that the continuity conditions on u, v and w all conspire to make $\beta_i(x)$ continuous on S_x. Lastly we compute

$$\begin{aligned}
\beta_i(x)(st) &= \overline{\pi}_x(u_{st}^* w_{st} v_{st}^*) = \overline{\pi}_x(\overline{\alpha}_s(u_t^*) u_s^* w_s w_t v_t^* v_s^*) \\
&= \overline{\pi}_x(v_s u_t^* v_s^* u_s^* w_s v_s^* v_s w_t v_t^* v_s^*) \\
&= \overline{\pi}_x(v_s u_t^* v_s^*) \beta_i(x)(s) \overline{\pi}_x(v_s w_t v_t^* v_s^*) \\
&= \beta_i(x)(s) \overline{\pi}_x(v_s u_t^* w_t v_t^* v_s^*) \\
&= \beta_i(x)(s) \overline{\pi}_x(v_s) \beta_i(x)(t) \overline{\pi}_x(v_s^*) \\
&= \beta_i(x)(s) \beta_i(x)(t).
\end{aligned}$$

Since $\beta_i(x)$ is a continuous $\mathbb{T}$-valued homomorphism, it is an element of $\widehat{S}_x$ and thus β_i is a section of $\widehat{S}$ on U_i. Now suppose $\omega \in \widehat{S}_x$. Then we compute for $f \in \Gamma_c(S, p^*\mathcal{A})$

$$\pi_x \rtimes (\omega \overline{\pi}_x(w))(\phi(f)) = \int_S \pi_x(\phi(f)(s)) \omega(s) \overline{\pi}_x(w_s) d\beta^x(s) \tag{5.28}$$

$$
\begin{aligned}
&= \int_S \pi_x(f(s))\omega(s)\overline{\pi}_x(u_s^* w_s)\,d\beta^x(x) \\
&= \int_S \pi_x(f(s))\omega(s)\beta_i(x)(s)\overline{\pi}_x(v_s)\,d\beta^x(s) \\
&= \pi_x \rtimes (\omega\beta_i(x)\overline{\pi}_x(v))(f).
\end{aligned}
$$

It follows from (5.28) that

$$
\begin{aligned}
\phi_i \circ \Phi \circ \psi_i^{-1}(\omega) &= \phi_i(\pi_x \rtimes (\omega\overline{\pi}_x(w)) \circ \phi) \\
&= \phi_i(\pi_x \rtimes (\omega\beta_i(x)\overline{\pi}_x(v))) \\
&= \omega\beta_i(x).
\end{aligned}
$$

Therefore β_i implements Φ on trivializations. Furthermore since ϕ_i, Φ, and ψ_i are all continuous, it is now straightforward to show that β_i is a continuous section. Thus $(\mathcal{U}, \Phi, \beta)$ is an $\widehat{S}$-bundle isomorphism of $(A \rtimes_\beta S)^\wedge$ onto $(A \rtimes_\alpha S)^\wedge$.

Now suppose that $(\mathcal{U}, \Phi, \beta)$ is an $\widehat{S}$-bundle isomorphism of $(A \rtimes_\alpha S)^\wedge$ onto $(A \rtimes_\beta S)^\wedge$. Let w^i and v^i implement α and β, respectively. Notice that $\mathcal{U}$ must be a common refinement of the local trivializing cover for α and β so that we may as well assume w^i and v^i are defined on $\mathcal{U}$. Fix $U_i \in \mathcal{U}$ and $x \in U_i$. For each $s \in S_x$ we define a unitary $u_s \in UM(A(x))$ by

$$
u_s := \beta_i(x)(s)w_s^i(v_s^i)^*. \tag{5.29}
$$

We need to show that (5.29) doesn't depend on the choice of U_i. So suppose $x \in U_j$ as well. (Notice we are going to have to keep track of the i and j for a while.) Let γ_{ij} and η_{ij} be the transition maps for $(A \rtimes_\alpha S)^\wedge$ and $(A \rtimes_\beta S)^\wedge$ respectively. Recall that we can view x as an element of $\widehat{S}_x$. Using the general theory of principal bundles, we obtain

$$
\begin{aligned}
\beta_i(x)\gamma_{ij}(x) &= \psi_i(\Phi(\phi_i^{-1}(\phi_i(\phi_j^{-1}(x))))) \\
&= \psi_i(\psi_j^{-1}(\psi_j(\Phi(\phi_j^{-1}(x))))) \\
&= \eta_{ij}(x)\beta_j(x).
\end{aligned}
$$

We use this fact to compute

$$
\begin{aligned}
\beta_i(x)(s)\overline{\pi}_x(w_s^i(v_s^i)^*) &= \beta_i(x)(s)\overline{\pi}_x(w_s^i(v_s^i)^* v_s^j(v_s^j)^*) \\
&= \beta_i(x)(s)\overline{\pi}_x(w_s^i)\overline{\pi}_x((v_s^i)^* v_s^j)\overline{\pi}_x((v_s^j)^*) \\
&= \beta_i(x)(s)\gamma_{ij}(x)(s)\overline{\pi}_x(w_s^i(v_s^j)^*) \\
&= \beta_j(x)(s)\eta_{ij}(x)(s)\overline{\pi}_x(w_s^i(v_s^j)^*)
\end{aligned} \tag{5.30}
$$

$$
\begin{aligned}
&= \beta_j(x)(s)\overline{\pi}_x(w_s^i((w_s^i)^* w_s^j)(v_s^j)^*) \\
&= \beta_j(x)(s)\overline{\pi}_x(w_s^j(v_s^j)^*).
\end{aligned}
$$

Remark 5.60. Since A has Hausdorff spectrum, each fibre $A(x)$ is simple [RW98, Lemma 5.1]. Thus, the only proper closed ideal is trivial, and π_x must be a faithful representation. It is then straightforward to calculate that the extension $\overline{\pi}_x$ is also faithful.

Hence (5.30) implies that

$$
\beta_i(x)(s)w_s^i(v_s^i)^* = \beta_j(x)(s)w_s^j(v_s^j)^*
$$

and that u_s is well defined. We now show that the u_s implement an equivalence between α and β. Observe that the first condition of Definition 5.34 is satisfied by construction. Dropping the i's again for convenience, we compute for $s, t \in S_x$

$$
\begin{aligned}
\overline{\pi}(u_{st}) &= \beta_i(x)(st)\overline{\pi}_x(w_{st}v_{st}^*) \\
&= \beta_i(x)(s)\beta_i(x)(t)\overline{\pi}_x(w_s v_s^*)\overline{\pi}_x(v_s w_t v_t^* v_s^*) \\
&= \overline{\pi}_x(u_s v_s u_t v_s^*) \\
&= \overline{\pi}_x(u_s \overline{\alpha}_s(u_t)).
\end{aligned}
$$

Again using the fact that π is faithful, this is sufficient to verify the second condition of equivalence. The continuity condition is straightforward to prove using the fact that the actions u and v are continuous, as well as the fact that β_i is a continuous section. For the last condition observe that for $a \in A(x)$

$$
\begin{aligned}
\pi(\operatorname{Ad} u_s(\alpha_s(a))) &= \pi(u_s v_s a v_s^* u_s^*) \\
&= \beta_i(x)(s)\overline{\beta_i(x)(s)}\pi(w_s v_s^* v_s a v_s^* v_s w_s^*) \\
&= \pi(w_s a w_s^*) = \pi(\beta_s(a)).
\end{aligned}
$$

Therefore $\operatorname{Ad} u_s \circ \alpha_s = \beta_s$ and $\{u_s\}$ implements an equivalence between α and β. □

Of course, this leads to the following corollary.

Corollary 5.61. *A locally unitary action of a continuously varying abelian group bundle S on a C^*-algebra with Hausdorff spectrum $S^{(0)}$ is determined, up to exterior equivalence, by the associated cohomological invariant of $(A \rtimes S)^\wedge$ as a principal $\widehat{S}$-bundle. Furthermore, the isomorphism class of $A \rtimes S$ is characterized by this invariant.*

Proof. This corollary just puts together Theorem 5.59 with Theorem 2.16 and Proposition 5.37. □

Remark 5.62. Since non-exterior equivalent actions can still have isomorphic crossed products, it is possible that $A \rtimes_\alpha S$ can be isomorphic to $A \rtimes_\beta S$ even though their invariants are different.

The last piece of the puzzle will be to prove that locally unitary actions are about as abundant as they can be. In other words we will show that every principal bundle can be obtained through a locally unitary action.

Theorem 5.63. *Suppose S is an abelian, second countable, locally compact Hausdorff, continuously varying group bundle and $q : X \to S^{(0)}$ is a principal S-bundle. Then the transformation groupoid C^*-algebra $C^*(S, X)$ has Hausdorff spectrum $S^{(0)}$ and the dual action of $\widehat{S}$ on $C^*(S, X)$ defined for $\omega \in \widehat{S}_u$ on $C_c(S_u \times q^{-1}(u))$ by*

$$\widehat{\mathrm{lt}}_\omega(f)(s, x) = \omega(s)f(s, x) \tag{5.31}$$

is locally unitary. Furthermore, $(C^*(S, X) \rtimes \widehat{S})^\wedge$ and X are isomorphic S-bundles.

Of course, before we can get down to the details of Theorem 5.63 we have a number of things to check.

Lemma 5.64. *Suppose S is a continuously varying abelian group bundle which acts continuously on a locally compact Hausdorff space X. Let q be the range map on X and define $X_u := q^{-1}(u)$ for all $u \in S^{(0)}$. Then $C^*(S, X)$ is a $C_0(S^{(0)})$-algebra and for each $u \in S^{(0)}$ restriction factors to an isomorphism from $C^*(S, X)(u)$ onto $C^*(S_u, X_u)$.*

Remark 5.65. Of course, now that we have a space called X floating around we can no longer denote elements of $S^{(0)}$ by x. Unfortunately, u is still going to conflict with our unitary notation, but we will make the best of it. To make matters worse, we are going to need to use ω to denote characters, which will look ugly when paired with the w's that turn up. It is moments like this which make the author wish the alphabet were longer. [Seu55]

Proof. Let $S \ltimes X$ be the transformation groupoid associated to S and X and recall that by definition $C^*(S, X)$ is equal to $C^*(S \ltimes X)$. Furthermore, recall that Proposition 4.38 states that the map $\Phi : C^*(S, X) \to C_0(X) \rtimes_{\mathrm{lt}} S$ defined on $C_c(S \ltimes X)$ by $\Phi(f)(s)(x) = f(s, x)$ is an isomorphism. Let $\mathcal{C}$ be the upper semicontinuous bundle associated to $C_0(X)$ and recall from Example 3.17 that $\mathcal{C}$ has fibres $C_0(X)(u) = C_0(X_u)$. Proposition 4.28 says that $C_0(X) \rtimes S$ is a $C_0(S^{(0)})$-algebra with the action defined for $\phi \in C_0(S^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{C})$ by

$$\phi \cdot f(s)(x) = \phi(p(s))f(s)(x).$$

Therefore we can use the isomorphism Φ to give $C^*(S,X)$ a $C_0(S^{(0)})$-algebra structure defined via

$$\phi \cdot f(s,x) = \Phi^{-1}(\phi \cdot \Phi(f))(s,x) = \phi(p(s))\Phi(f)(s)(x) = \phi(p(s))f(s,x).$$

Now fix $u \in S^{(0)}$. By construction Φ is $C_0(X)$-linear so that Φ factors to an isomorphism $\bar{\Phi} : C^*(S,X)(u) \to C_0(X) \rtimes S(u)$. The restriction map ρ factors to an isomorphism $\bar{\rho}$ of $C_0(X) \rtimes S(u)$ with $C_0(X_u) \rtimes S_u$. Furthermore since the action of S_u on $C_0(X_u)$ is still given by left translation, there is another isomorphism $\Psi : C_0(X_u) \rtimes S_u \to C^*(S_u, X_u)$ defined in the same manner as Φ. Thus, we get an isomorphism $\Psi \circ \bar{\rho} \circ \bar{\Phi}$ of $C^*(S,X)(u)$ onto $C^*(S_u, X_u)$. We would like to see that this isomorphism is given by restriction for $f \in C_c(S \ltimes X)$. Let σ be the restriction map from $C_c(S \ltimes X)$ onto $C_c(S_u \times X_u)$ and (foolishly) let q denote both the quotient map from $C^*(S,X)$ onto $C^*(S,X)(u)$ and the quotient map from $C_0(X) \rtimes S$ onto $C_0(X) \rtimes S(u)$. We then have for $f \in C_c(S \ltimes X)$

$$\begin{aligned}
\Psi \circ \bar{\rho} \circ \bar{\Phi}(q(f))(s,x) &= \bar{\rho}(\bar{\Phi}(q(f)))(s)(x) = \bar{\rho}(q(\Phi(f)))(s)(x) \\
&= \rho(\Phi(f))(s)(x) = \Phi(f)(s)(x) \\
&= f(s,x) = \sigma(f)(s,x).
\end{aligned}$$

Therefore, the isomorphism from $C^*(S,X)(u)$ onto $C^*(S_u, X_u)$ is just the factorization of the restriction map on $C_c(S \ltimes X)$. □

Remark 5.66. Propositions like the one above are really just notational trickery. Philosophically, $C^*(S,X)$ and $C_0(X) \rtimes S$ are (basically) completions of the same function algebra and should be treated as the same object.

Now, in order for Theorem 5.63 to work we need to know that $C^*(S,X)$ has Hausdorff spectrum.

Proposition 5.67. *Suppose S is a continuously varying abelian group bundle and that $q : X \to S^{(0)}$ is a principal S-bundle. Then the transformation groupoid algebra $C^*(S,X)$ has Hausdorff spectrum $S^{(0)}$. Furthermore the fibre of $C^*(S,X)$ over $u \in S^{(0)}$ is $C^*(S_u, X_u)$ where $X_u = q^{-1}(u)$.*

Proof. Using Lemma 5.64 we conclude that $C^*(S,X)$ is a $C_0(S^{(0)})$-algebra and that restriction factors to an isomorphism of $C^*(S,X)(u)$ with $C^*(S_u, X_u)$. Since $C^*(S,X)$ is a $C^*(S^{(0)})$-algebra there is a continuous surjection q of $C^*(S,X)^\wedge$ onto $S^{(0)}$. Furthermore we identify $q^{-1}(u)$ with $C^*(S_u, X_u)^\wedge$ in the usual fashion. Next, let $\phi : X_u \to S_u$ be the restriction of one of the trivializing maps for X to X_u. Since ϕ is a homeomorphism we can pull back the group structure from S_u to X_u and turn ϕ into a group isomorphism. Furthermore it follows from Proposition 2.18 that ϕ is equivariant with

respect to the action of S_u on X_u. Therefore

$$s \cdot \phi^{-1}(t) = \phi^{-1}(st)$$

so that if we identify X_u with S_u then the action of S_u on X_u becomes the action of S_u on itself by translation. In other words $C^*(S_u, X_u)$ is isomorphic to $C^*(S_u, S_u)$. It follows from the von Neumann Theorem, Corollary 4.42, that $C^*(S_u, S_u)$ is isomorphic to the compact operators on some separable Hilbert space. Hence $C^*(S_u, S_u)$, and therefore $C^*(S, X)(u)$, has a unique irreducible representation. It follows that the map q is injective.

All that remains is to show that q is open, or equivalently, closed. Suppose C is a closed subset of $C^*(S, X)^\wedge$. Then there is some ideal I such that $C = \{\pi \in C^*(S, X)^\wedge : I \subset \ker \pi\}$. Let $D = \{u \in S^{(0)} : I \subset I_u\}$ where the ideal I_u in $C^*(S, X)$ is given by

$$I_u = \overline{\operatorname{span}}\{\phi \cdot f : \phi \in C_0(S^{(0)}), f \in C_c(S \ltimes X), \phi(u) = 0\}.$$

We claim that $D = q(C)$. If $u \in D$ and $\pi = q^{-1}(u)$ then π factors to a faithful representation of $C^*(S_u, X_u)$ so that $\ker \pi = I_u$. Thus $I \subset I_u = \ker \pi$ and $\pi \in C$. Conversely if $\pi \in C$ and $u = q(\pi)$ then π factors to a faithful representation of $C^*(S_u, X_u)$ so that $I_u = \ker \pi$. It follows that $I \subset \ker \pi = I_u$ and $u \in D$. All that is left is to show that D is closed. Suppose $u_i \to u$ in $S^{(0)}$ and $u_i \in D$ for all i. Then since $I \subset I_{u_i}$ for all i we have $f(u_i) = 0$ for all $f \in I$. However, f is continuous when viewed as a function on $S^{(0)}$ so that $f(u) = 0$. Thus $f \in I_u$ and $u \in D$. □

Next, we show that there is a dual action of $\widehat{S}$ on $C^*(S, X)$ induced by left translation. Since it isn't much harder, we actually prove this result in greater generality. Unfortunately, we can't just jump right in. Verifying the continuity condition will take work. In particular we have to deal with the topology associated to the crossed bundle product $A \rtimes S$.

Lemma 5.68. *Suppose (A, S, α) is a separable dynamical system and that S is an abelian group bundle. Let $\mathcal{A}$ be the upper-semicontinuous bundle associated to A, and define*

$$\widehat{S} * S = \{(\omega, s) \in \widehat{S} \times S : \hat{p}(\omega) = p(s)\},$$

and let $p : \widehat{S} * S \to S^{(0)}$ be given by $p(\omega, s) = p(s)$. Then there is a map $\iota : \Gamma_c(\widehat{S} * S, p^*\mathcal{A}) \to \hat{p}^*(A \rtimes S)$ such that

$$\iota(f)(\omega)(s) = f(\omega, s).$$

Furthermore ι is continuous with respect to the inductive limit topology and the range of ι is dense.

Proof. First observe that, since f is continuous and compactly supported, $\iota(f)(\omega)$ will be a continuous, compactly supported function from $S_{\hat{p}(\omega)}$ into $A(\hat{p}(\omega))$. Thus $\iota(f)(\omega) \in C_c(S_{\hat{p}(\omega)}, A(\hat{p}(\omega)))$ and $\iota(f)$ is a section of $\hat{p}^*(A \rtimes S)$. Furthermore, it is clear that $\iota(f)$ has compact support. We would like to show that $\iota(f)$ is continuous as a function into $\mathcal{E}$ where $\mathcal{E}$ is the upper-semicontinuous bundle associated to $A \rtimes S$.

We start out with a simpler function. Suppose $g \in C_c(\widehat{S})$, $h \in C_c(S)$ and $a \in A$. Define $g \otimes h \otimes a$ on $\widehat{S} * S$ by $g \otimes h \otimes a(\omega, s) = g(\omega)h(s)a(p(s))$. It is clear that $g \otimes h \otimes a \in \Gamma_c(\widehat{S} * S, p^*\mathcal{A})$. Furthermore if we view $h \otimes a$ as an element of $\Gamma_c(S, p^*\mathcal{A})$ then $\iota(g \otimes h \otimes a)(\omega) = g(\omega)(h \otimes a)(\hat{p}(\omega))$ where $(h \otimes a)(\hat{p}(\omega))$ is just the restriction of $h \otimes a$ to $S_{\hat{p}(\omega)}$. Since $h \otimes a$ defines a continuous section of $\mathcal{E}$ it is easy to see that $\iota(g \otimes h \otimes a)$ is a continuous function from $\widehat{S}$ into $\mathcal{E}$. Thus $\iota(g \otimes h \otimes a) \in \Gamma_c(\widehat{S}, \hat{p}^*\mathcal{E})$.

We now show ι preserves convergence with respect to the inductive limit topology. Suppose $f_i \to f$ uniformly in $\Gamma_c(\widehat{S} * S, p^*\mathcal{A})$ and that eventually the supports are contained in some fixed compact set K. Clearly the supports of $\iota(f)$ are eventually contained in the projection of K to $\widehat{S}$. Fix $\epsilon > 0$ and let M be an upper bound for $\{\beta^u(L)\}$ where L is the projection of K to S. Then eventually $\|f_i - f\|_\infty < \epsilon/L$. Thus for large i we have, for $\omega \in \widehat{S}_u$ and making use of the fact that S_u is abelian so the I-norm on $C_c(S_u, A(u))$ only has one term,

$$\begin{aligned}
\|\iota(f_i)(\omega) - \iota(f)(\omega)\| &\leq \|\iota(f_i)(\omega) - \iota(f)(\omega)\|_I \\
&= \int_S \|f_i(\omega, s) - f(\omega, s)\| d\beta^u(s) \\
&\leq \|f_i - f\|_\infty L < \epsilon
\end{aligned}$$

Thus $\iota(f_i) \to \iota(f)$ uniformly and hence with respect to the inductive limit topology.

Now suppose $f \in \Gamma_c(\widehat{S} * S, p^*\mathcal{A})$ and that $\omega_i \to \omega$ in $\widehat{S}$. Fix $\epsilon > 0$ and let U and V be relatively compact neighborhoods of the projections of $\operatorname{supp} f$ to $\widehat{S}$ and S respectively. Since f is a section of a pull back bundle we can use Proposition 3.40 to find $\{g_i\}_{i=1}^N \in C_c(\widehat{S} * S)$ and $\{a_i\}_{i=1}^N \in A$ such that $\|f - \sum_i g_i \otimes a_i\|_\infty < \epsilon/2$. For each $1 \leq i \leq N$ extend g_i to all of $C_c(\widehat{S} \times S)$ and choose $h_i^j \in C_c(\widehat{S})$ and $k_i^j \in C_c(S)$ such that $\|g_i - \sum_j h_i^j \otimes k_i^j\|_\infty < \epsilon/(2N\|a_i\|)$. It then follows from some simple computations that

$$\begin{aligned}
\left\| f - \sum_{i=1}^N \sum_j h_i^j \otimes k_i^j \otimes a_i \right\|_\infty &\leq \left\| f - \sum_{i=1}^N g_i \otimes a_i \right\|_\infty \\
&\quad + \left\| \sum_{i=1}^N g_i \otimes a_i - \sum_{i=1}^N \sum_j h_i^j \otimes k_i^j \otimes a_i \right\|_\infty
\end{aligned}$$

$$\leq \epsilon/2 + \sum_{i=1}^{N} \|a_i\| \left\| g_i - \sum_j h_i^j \otimes k_i^j \right\|_\infty < \epsilon.$$

Furthermore, we can multiply the h_i^j and k_i^j by functions which vanish off U and V respectively so that $\operatorname{supp} h_i^j \otimes k_i^j \otimes a \subset \overline{U} \times \overline{V}$. This construction shows that sums of elements of the form $h \otimes k \otimes a$ for $h \in C_c(\widehat{S})$, $k \in C_c(S)$ and $a \in A$ are dense in $\Gamma_c(\widehat{S} * S, p^*\mathcal{A})$ with respect to the inductive limit topology.

At last we can show that $\iota(f)$ is continuous for $f \in \Gamma_c(\widehat{S} * S, p^* S)$. Let $g_i = \sum_k h_i^k \otimes k_i^k \otimes a_i^k$ be a sequence converging to f in the inductive limit topology as above. Let $\omega_j \to \omega$ and fix $\epsilon > 0$. For some very large I we have $\|\iota(f) - \iota(g_I)\|_\infty < \epsilon$. In particular $\|\iota(f)(\omega) - \iota(g_I)(\omega)\| < \epsilon$ and $\|\iota(f)(\omega_j) - \iota(g_I)(\omega_j)\| < \epsilon$ for all i. Since sums of continuous functions are continuous, $\iota(g_I)(\omega_j) \to \iota(g_I)(\omega)$ and it follows from the last part of Proposition 3.2 that $\iota(f)$ is continuous.

Thus ι maps $\Gamma_c(\widehat{S} * S, p^*\mathcal{A})$ into $\hat{p}^*(A \rtimes S)$. The last thing we need to do is verify that ι has dense range. Given $\phi \in C_0(\widehat{S})$ and $f \in \Gamma_c(\widehat{S} * S, p^*\mathcal{A})$ we can define a new function $h \in \Gamma_c(\widehat{S} * S, p^*\mathcal{A})$ by $h(\omega, s) = \phi(\omega) h(\omega, s)$. It is easy enough to see that $\phi \cdot \iota(f) = \iota(h)$. Thus $\operatorname{ran} \iota$ is closed under the $C_0(\widehat{S})$ action. Now fix $\omega \in \widehat{S}_u$. Given $a \in A$ and $f \in C_c(S_u)$ extend f to a function $h \in C_c(S)$ and choose $g \in C_c(\widehat{S})$ so that $g(\omega) = 1$. Then clearly $\iota(g \otimes f \otimes a)(\omega) = h|_{S_u} \otimes a = f \otimes a$. Thus $\operatorname{ran} \iota$ contains sums of elementary tensors in $A(u) \rtimes S_u$. It follows from Proposition 3.11 that $\operatorname{ran} \iota$ is dense. □

The following corollary isn't necessary to build the dual action, but it will be needed in the proof of Theorem 5.63 so we include it here.

Corollary 5.69. *Suppose S is an abelian, second countable, locally compact Hausdorff continuously varying group bundle and $q : X \to S^{(0)}$ is a principal S-bundle. Define*

$$\widehat{S} * S * X := \{(\omega, s, x) \in \widehat{S} \times S \times X : \hat{p}(\omega) = p(s) = q(x)\}$$

Then there is a map $\iota : C_c(\widehat{S} * S * X) \to \hat{p}^* C^*(S, X)$ such that

$$\iota(f)(\omega)(s, x) = f(\omega, s, x).$$

Furthermore ι is continuous with respect to the inductive limit topology and the range of ι is dense.

Proof. Let $\mathcal{C}$ be the upper-semicontinuous bundle associated to $C_0(X)$. Since both algebras can be viewed as completions of $C_c(S \ltimes X)$ we will use Proposition 4.38 to identify $C_0(X) \rtimes S$ with $C^*(S, X)$. In particular, we will not distinguish between

functions in $C_c(S \ltimes X)$ and their corresponding functions in $\Gamma_c(S, p^*\mathcal{C})$. Consider the map $\iota_1 : \Gamma_c(\widehat{S} * S, p^*\mathcal{C}) \to \hat{p}^* C^*(S, X)$ given by

$$\iota_1(f)(\omega)(s, x) = \iota_1(\omega)(s)(x) := f(\omega, s)(x).$$

It follows from Lemma 5.68 that this map is continuous with respect to the inductive limit topology and its range is dense in $\hat{p}^* C^*(S, X)$. Now consider the map $\iota_2 : C_c(\widehat{S} * S * X) \to \Gamma_c(\widehat{S} * S, p^*\mathcal{C})$ given by $\iota_2(f)(\omega, s)(x) = f(\omega, s, x)$. It follows from Lemma 4.37 that ι_2 is surjective and preserves the inductive limit topology. Thus the map $\iota = \iota_2 \circ \iota_1$ has the correct form and all the right properties. $\square$

Now we can finally tackle the dual action construction. This will provide the last tool we need to demonstrate Theorem 5.63.

Proposition 5.70. *Suppose (A, S, α) is a separable dynamical system and that S is an abelian group bundle. Then for each $\omega \in \widehat{S}$ there is an automorphism $\hat{\alpha}_\omega$ on $A \rtimes S(\hat{p}(\omega))$ defined for $f \in C_c(S_{\hat{p}(\omega)}, A(\hat{p}(\omega)))$ by*

$$\hat{\alpha}_\omega(f)(s) = \overline{\omega(s)} f(s).$$

With this action $(A \rtimes S, \widehat{S}, \hat{\alpha})$ is a dynamical system.

Proof. Recall that $A \rtimes S$ is a $C_0(S^{(0)})$-algebra with restriction factoring to an isomorphism of $A \rtimes S(u)$ with $A(u) \rtimes S_u$. Fix $u \in S^{(0)}$, $\omega \in \widehat{S}_u$ and define α_ω as above. Since $\omega(s)$ is unimodular for all $s \in S$, it follows from a simple calculation that α_ω is continuous with respect to the inductive limit topology. Furthermore, we have for $f, g \in C_c(S_u, A(u))$

$$\begin{aligned}
\alpha_\omega(f) * \alpha_\omega(g)(s) &= \int_S \overline{\omega(t)} f(t) \alpha_t(\overline{\omega(t^{-1}s)} g(t^{-1}s)) d\beta^u(t) \\
&= \overline{\omega(s)} f * g(s) = \alpha_\omega(f * g)(s), \quad \text{and} \\
\alpha_\omega(f)^*(s) &= \alpha_s(\omega(s^{-1}) f(s^{-1})^*) \\
&= \overline{\omega(s)} f^*(s) = \alpha_\omega(f^*)(s).
\end{aligned}$$

Thus α_ω is a $*$-homomorphism and it follows from Corollary 3.134 that α_ω extends to a map from $A(u) \rtimes S_u$ into itself. Furthermore observe that $\alpha_u = \mathrm{id}$ and

$$\alpha_\omega(\alpha_\chi(f))(s) = \overline{\omega(s)\chi(s)} f(s) = \alpha_{\omega\chi}(f)(s).$$

It follows that, one, α_ω is an automorphism, and, two, that α respects the groupoid structure. All that remains is to verify the continuity condition.

Let $\mathcal{E}$ be the bundle associated to $A \rtimes S$ and suppose $\omega_i \to \omega$ in $\widehat{S}$ and $f_i \to f$ in $\mathcal{E}$. Now choose $g \in \hat{p}^*(A \rtimes S)$ such that $g(\omega) = f$. It follows from Lemma 5.64 that we can view $\Gamma_c(\widehat{S} * S, p^*\mathcal{A})$ as lying inside $\hat{p}^*(A \rtimes S)$ and can choose $h \in \Gamma_c(\widehat{S} * S, p^*\mathcal{A})$ such that $\|h - g\|_\infty < \epsilon/2$. Define $\alpha(h)(\omega, s) = \overline{\omega(s)} h(\omega, s)$. It is clear that $\alpha(h) \in \Gamma_c(\widehat{S} * S, p^*\mathcal{A})$. It is also easy to see that $\alpha(h)(\omega) = \alpha_\omega(h(\omega))$. Thus

$$\|\alpha(h)(\omega) - \alpha_\omega(f)\| = \|\alpha_\omega(h(\omega) - g(\omega))\| < \epsilon/2 < \epsilon.$$

Next, since $g(\omega_i) \to g(\omega) = f$ and $f_i \to f$ we have $\|g(\omega_i) - f_i\| \to 0$. Therefore, eventually, we have

$$\begin{aligned} \|\alpha(h)(\omega_i) - \alpha_{\omega_i}(f_i)\| &\le \|\alpha_{\omega_i}(h(\omega_i) - g(\omega_i))\| + \|\alpha_{\omega_i}(g(\omega_i) - f_i)\| \\ &\le \epsilon/2 + \|g(\omega_i) - f_i\| < \epsilon. \end{aligned}$$

It follows from Proposition 3.2 that $\alpha_{\omega_i}(f_i) \to \alpha_\omega(f)$. □

Remark 5.71. The action from Proposition 5.70 is a generalization of the usual Takai dual action for abelian groups [Wil07, Section 7]. In particular there is a Takai Duality Theorem for abelian group bundles which states that $(A \rtimes_\alpha S) \rtimes_{\hat{\alpha}} \widehat{S}$ is isomorphic to $A \otimes_{C_0(X)} \mathcal{K}(\mathcal{Z})$ where $\mathcal{Z}$ is the Hilbert $C_0(S^{(0)})$-bundle from Theorem 4.41. However, this theorem is really nothing more than a "bundled" version of the theorem for groups. Although it takes some work, the proof boils down to observing that the isomorphism given in [Wil07, Section 7] respects the total space structure. Since there is no interesting "groupoid" component to this result, it has been omitted.

We are now ready to prove our existence theorem.

Proof of Theorem 5.63. We have shown in Proposition 5.67 that $C^*(S, X)$ has Hausdorff spectrum $S^{(0)}$ and that restriction factors to an isomorphism of $C^*(S, X)(u)$ with $C^*(S_u, X_u)$ where $X_u = q^{-1}(u)$. Furthermore we showed in Proposition 5.70 that there is an action of $\widehat{S}$ which, after making the usual identification of $C^*(S, X)$ with $C_0(X) \rtimes_{\mathrm{lt}} S$, is given by

$$\widehat{\mathrm{lt}}_\omega(f)(s, x) = \overline{\omega(s)} f(s, x)$$

for $f \in C_c(S_u \times X_u)$. We need to show that $\widehat{\mathrm{lt}}$ is locally unitary. Let $\mathcal{U}$ be a trivializing cover of X and let ϕ_i be the local trivializations. Fix $U_i \in \mathcal{U}$. Then for all $w \in U_i$, $\omega \in \widehat{S}_w$ and $f \in C_c(S_w \times X_w)$ define

$$u_\omega f(s, x) := \overline{\omega(\phi_i(x))} f(s, x). \tag{5.32}$$

Viewing w as the trivial element[7] in $\widehat{S}_w$ we clearly have $u_w = \mathrm{id}$. Furthermore, given $\omega, \chi \in \widehat{S}_w$,

$$u_{\omega\chi} f(s,x) = \omega(\phi_i(x))\chi(\phi_i(x))f(s,x) = u_\omega u_\chi f(s,x).$$

Thus u is a homomorphism on S_w. Next we will show that u is adjointable. Recall that we equip $C^*(S_w, X_w)$ with the inner product $\langle f, g\rangle = f^* * g$. For all $f, g \in C_c(S_w \times X_w)$ we have

$$\begin{aligned}
\langle u_\omega f, g\rangle(s,x) &= (u_\omega f)^* * g(s,x) \\
&= \int_S \overline{u_\omega f(t^{-1}, t^{-1}\cdot x)} g(t^{-1}s, t^{-1}\cdot x) d\beta^w(t) \\
&= \int_S \omega(\phi_i(t^{-1}\cdot x))\overline{f(t^{-1}, t^{-1}\cdot x)} g(t^{-1}s, t^{-1}\cdot x) d\beta^w(t) \\
&= \int_S \overline{f(t^{-1}, t^{-1}\cdot x)} u_{\omega^{-1}} g(t^{-1}s, t^{-1}\cdot x) d\beta^w(t) \\
&= f^* * (u_{\omega^{-1}} g)(s,x) = \langle f, u_{\omega^{-1}} g\rangle(s,x).
\end{aligned}$$

This shows that u_ω is adjointable on $C_c(S_w, X_w)$ and we can also observe that

$$\begin{aligned}
\|u_\omega f\|^2 &= \|\langle u_\omega f, u_\omega f\rangle\| = \|\langle f, u_{\omega^{-1}} u_\omega f\rangle\| \\
&= \|\langle f, f\rangle\| = \|f\|^2.
\end{aligned}$$

Thus u_ω is isometric on $C_c(S_w, X_w)$ and as such it can be extended to an operator on $C^*(S_w, X_w)$. Since u_ω is adjointable on a dense subspace with $u_\omega^* = u_{\omega^{-1}} = u_\omega^{-1}$, we know u_ω is a unitary multiplier on $C^*(S_w, E_w)$. Consider the collection $\{u_\omega\}_{\omega\in\hat{p}^{-1}(U_i)}$. All that remains for $\{u_\omega\}$ to define a unitary action of $\hat{p}^{-1}(U_i)$ on $C^*(S,X)(U_i)$ is continuity.

Let $\mathcal{E}$ be the bundle associated to $C^*(S,X)$ and fix $\epsilon > 0$. Suppose $\omega_j \to \omega$ in $\hat{p}^{-1}(U_i)$ and $f_j \to f$ in $\mathcal{E}|_{U_i}$. Choose $g \in \hat{p}^* C^*(S,X)$ such that $g(\omega) = f$. Using Corollary 5.69 we can find a continuous, compactly supported function h on $\widehat{S} * S * X$ such that $\|\iota(h) - g\|_\infty < \epsilon/2$. Consider the open set $O = \hat{p}^{-1}(U_i) * p^{-1}(U_i) * q^{-1}(U_i)$ in $\widehat{S} * S * X$. We define a new function $k \in C(O)$ by

$$k(\chi, s, x) = \psi(p(s))\overline{\chi(\phi_i(x))} h(\chi, s, x)$$

where $\psi \in C_c(U_i)$ is some function which is one on a neighborhood of $\hat{p}(\omega)$. Now k is clearly compactly supported with $\operatorname{supp} k \subset \hat{p}^{-1}(\operatorname{supp}\psi) * p^{-1}(\operatorname{supp}\psi) * q^{-1}(\operatorname{supp}\psi) \subset O$. Therefore we can, and do, extend k by zero to all of $\widehat{S} * S * X$. Next we observe

[7] It even looks like it should be a character! ($\omega \approx w$)

the following facts. First,

$$\iota(k)(\omega)(s,x) = \overline{\omega(\phi_i(x))}h(\omega,s,x) = u_\omega \iota(h)(\omega)(s,x).$$

In a similar fashion we see that eventually $\iota(k)(\omega_j) = u_{\omega_j}\iota(h)(\omega_j)$. Second, observe that

$$\|\iota(k)(\omega) - u_\omega f\| = \|u_\omega(\iota(h)(\omega) - g(\omega))\| = \|\iota(h)(\omega) - g(\omega)\| < \epsilon/2.$$

Furthermore $f_i \to f$ and $g(\omega_i) \to g(\omega) = f$ so that $\|f_i - g(\omega_i)\| \to 0$. Thus, eventually, we have

$$\begin{aligned}\|\iota(k)(\omega_i) - u_{\omega_i} f_i\| &\leq \|u_{\omega_i}(\iota(h)(\omega_i) - g(\omega_i))\| + \|u_{\omega_i}(g(\omega_i) - f_i)\| \\ &\leq \epsilon/2 + \|g(\omega_i) - f_i\| < \epsilon.\end{aligned}$$

Finally we observe that $\iota(k)(\omega_i) \to \iota(k)(\omega)$ since $\iota(k)$ is a continuous section. It follows that $u_{\omega_i} f_i \to u_\omega f$ and that $\{u_\omega\}$ defines a unitary action of $\hat{p}^{-1}(U_i)$ on $C^*(S,X)(U_i)$.

Next, we show that u implements $\widehat{\mathrm{lt}}$ on $\hat{p}^{-1}(U_i)$. Suppose $u \in U_i$ and $\omega \in \widehat{S}_u$. Then for $f \in C_c(S_u \times X_u)$ we have

$$\begin{aligned}u_\omega f u_\omega^*(s,x) &= \overline{\omega(\phi_i(x))}(u_\omega f^*)^*(s,x) = \overline{\omega(\phi_i(x)) u_\omega f^*(s^{-1}, s^{-1}\cdot x)} \\ &= \overline{\omega(\phi_i(x))}\omega(\phi_i(s^{-1}\cdot x))f(s,x) = \overline{\omega(\phi_i(x))\omega(s)}\omega(\phi_i(x))f(s,x) \\ &= \overline{\omega(s)}f(s,x) = \widehat{\mathrm{lt}} f(s,x)\end{aligned}$$

where we have used the fact that ϕ_i is equivariant with respect to the action of S on X. Thus $\widehat{\mathrm{lt}}$ is unitarily implemented on $\hat{p}^{-1}(U_i)$. Since we performed this construction for each element of the cover $\mathcal{U}$ it follows that $\widehat{\mathrm{lt}}$ is locally unitary.

Consider $Y = (C^*(S,X) \rtimes \widehat{S})^\wedge$. Now, Y is a principal $\widehat{\widehat{S}}$-bundle and in light of Theorem 2.52 a principal S-bundle as well. We would like to show that Y is isomorphic to X. Using Theorem 2.16 it suffices to show that X and Y have the same cohomological invariant. Let γ_{ij} be the transition functions for X with respect to the trivializing maps ϕ_i. Let η_{ij} be the transition functions for Y and recall from Theorem 5.58 that for $v \in U_{ij}$ and $\omega \in \widehat{S}_v$

$$\eta_{ij}(v)(\omega) = \overline{\pi}_v((u_\omega^i)^* u_\omega^j)$$

where π_v is the unique irreducible representation of $A(v)$ and u_ω^i, u_ω^j are the unitaries constructed above. Recall that $\phi_i \circ \phi_j^{-1}(s) = \gamma_{ij}(p(s))s$ so that, letting $x = \phi_j^{-1}(s)$,

we have $\phi_i(x) = \gamma_{ij}(q(x))\phi_j(x)$. We now compute for $f \in C_c(S_v, X_v)$

$$\begin{aligned}((u^i_\omega)^* u^j_\omega f)(s,x) &= \overline{\omega^{-1}(\phi_i(x))\omega(\phi_j(x))} f(s,x) \\ &= \omega(\gamma_{ij}(v)\phi_j(x))\overline{\omega(\phi_j(x))} f(s,x) \\ &= \omega(\gamma_{ij}(v)) f(s,x) \\ &= \hat{\gamma}_{ij}(v)(\omega) f(s,x)\end{aligned}$$

where $\hat{\gamma}_{ij}(v)$ denotes the image of $\gamma_{ij}(v)$ in the double dual. Therefore $(u^i_\omega)^* u^j_\omega = \hat{\gamma}_{ij}(v)(\omega)$ and, since π_v is faithful, $\eta_{ij}(v)(\omega) = \hat{\gamma}_{ij}(v)(\omega)$. Thus, once we identify S with $\widehat{\widehat{S}}$, the cohomological invariants of X and Y are identical. □

Example 5.72. Theorem 5.63 says that any principal S-bundle X gives rise to a locally unitary action of $\widehat{S}$ on $C^*(S, X)$ and in particular this holds for locally σ-trivial bundles. Thus Examples 2.29 and 2.30 yield examples of locally unitary actions.

Remark 5.73. It is worth describing, at least briefly, how this material generalizes [PR84]. Suppose H is an abelian group and A has Hausdorff spectrum X. If α is an action of H on A then, as in Example 4.7, we can form the transformation groupoid $H \ltimes X$ and there is an action β of $H \ltimes X$ on A. Furthermore we have $A \rtimes_\alpha X \cong A \rtimes_\beta (H \ltimes X)$. Without getting into the details, α is locally unitary according [PR84] if, for each $\pi \in X$, there is an open neighborhood U of π and a strictly continuous map $u : H \to M(A)$ such that for each $\rho \in U$, $\bar{\rho} \circ u$ is a representation of H on $\mathcal{H}_\rho$ which implements α. In particular this implies that $\rho \circ \alpha_s^{-1} = \bar{\rho}(u_s)\rho\bar{\rho}(u_s^*)$ is equivalent to ρ. Thus the action of H on X induced by α is trivial and the transformation groupoid $H \ltimes X$ is the trivial group bundle. What's more, since A has Hausdorff spectrum, it is not hard to show that $u_s(x)$ implements $\beta_{(s,x)}$ on $A(x)$ and that β is unitarily implemented by $\{u_s(x)\}$ on $H \times U$. Thus β is a locally unitary action of $H \times S$ on A. Now, the dual of $H \times X$ is $\widehat{H} \times X$ and we have, according to Theorem 5.58, that $(A \rtimes_\alpha X)^\wedge \cong (A \rtimes_\beta (H \times X))^\wedge$ is a principal $\widehat{H} \times X$ bundle. However, it follows from Proposition 2.22 that this implies $(A \rtimes_\alpha X)^\wedge$ is a principal $\widehat{H}$-bundle. From here it is straightforward to see how the results of this section are related to those in [PR84].

Chapter 6

Fine Structure of Groupoid Crossed Products

In this chapter we present the main results of the thesis. An important aspect of the proof is the induction process detailed in Section 6.1. This allows us to induce representations from the crossed product by any closed subgroupoid to the whole crossed product. We will use this in Section 6.2 to show that when the orbits are T_0 every irreducible representation of the crossed product is induced from a stabilizer subgroup. We use this result in Section 6.3 to identify the spectrum of $A \rtimes G$ with a quotient of the spectrum of $A \rtimes S$, where S is the stabilizer subgroupoid of G.

6.1 Induction

A key tool in our study of the representation theory of crossed products will be the ability to induce representations from closed subgroupoids. This notion has been around for groups since [Fro98], although this particular section is more closely modeled after [Wil07, Chapter 5]. Our starting point will be the imprimitivity groupoid from Section 1.2.1. Specifically, we begin by describing an action which arises from coupling the imprimitivity groupoid with a dynamical system.

Proposition 6.1. *Let (A, G, α) be a separable groupoid dynamical system and suppose H is a closed subgroupoid of G with a Haar system. Let $X = s^{-1}(H^{(0)})$ and let G^H be the associated imprimitivity groupoid. Define $\rho : X/H \to G^{(0)}$ via $\rho(\gamma \cdot H) = r(\gamma)$ and let ρ^*A be the pull back algebra. Then the collection $\sigma = \{\sigma_{[\gamma,\eta]}\}_{[\gamma,\eta]\in G^H}$ where $\sigma_{[\gamma,\eta]} : A(r(\eta)) \to A(r(\gamma))$ is defined for $a \in A(r(\gamma))$ by*

$$\sigma_{[\gamma,\eta]}(a) = \alpha_{\gamma\eta^{-1}}(a) \tag{6.1}$$

defines an action of G^H on ρ^*A.

Proof. It follows from Proposition 3.38 that the pull back ρ^*A is a $C_0(X/H)$-algebra with fibres $A(\gamma \cdot H) = A(r(\gamma))$. Recall that we identify $(G^H)^{(0)}$ with X/H so that we may also view ρ^*A as a $C_0((G^H)^{(0)})$-algebra. Next, we want to show that $\sigma_{[\gamma,\eta]}$ is independent of the choice of representatives γ and η. If $[\gamma,\eta] = [\gamma',\eta']$ then there exists $\zeta \in H$ such that $\gamma\zeta = \gamma'$ and $\eta\zeta = \eta'$. However this implies that $\gamma\eta^{-1} = \gamma'(\eta')^{-1}$ and that $\sigma_{[\gamma,\eta]}$ is well defined.

Moving on, it is clear that $\sigma_{[\gamma,\eta]} = \alpha_\gamma \circ \alpha_\eta^{-1}$ is an isomorphism of $A(s([\gamma,\eta])) = A(r(\eta))$ onto $A(r([\gamma,\eta])) = A(r(\gamma))$. Next, if $[\gamma,\eta],[\eta,\zeta] \in G^H$ and $a \in A(r(\zeta))$ then

$$\sigma_{[\gamma,\eta]} \circ \sigma_{[\eta,\zeta]}(a) = \alpha_{\gamma\eta^{-1}}(\alpha_{\eta\zeta^{-1}}(a)) = \alpha_{\gamma\zeta^{-1}}(a) = \sigma_{[\gamma,\zeta]}(a).$$

Thus the action respects the groupoid operations. Lastly we have to show that if $[\gamma_i,\eta_i] \to [\gamma,\eta]$ in G^H and $a_i \to a$ in $\mathcal{A}$ such that $p(a_i) = r(\eta_i)$ for all i and $p(a) = r(\eta)$ then

$$\sigma_{[\gamma_i,\eta_i]}(a_i) = \alpha_{\gamma_i\eta_i^{-1}}(a_i) \to \alpha_{\gamma\eta^{-1}}(a) = \sigma_{[\gamma,\eta]}(a). \tag{6.2}$$

After passing to a subnet it will suffice to show that a sub-subnet converges. However, we can pass to a subnet, relabel and choose new representatives so that $\gamma_i \to \gamma$ and $\eta_i \to \eta$. However, (6.2) now holds because α is a continuous action. □

Remark 6.2. The crossed product $\rho^*A \rtimes_\sigma G^H$ is the completion of $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$. As in Remark 3.36, elements of this function algebra can be viewed as continuous, compactly supported maps from G^H into $\mathcal{A}$ such that $f([\gamma,\eta]) \in A(r(\gamma))$ for all $[\gamma,\eta] \in G^H$.

Now we use the equivalence theorem to build an imprimitivity bimodule. This shows that up to Morita equivalence we really don't get anything new from σ, and that it is equivalent to the restriction of α to H.

Proposition 6.3. *Suppose (A,G,α) is a separable dynamical system and let λ be a Haar system for G. Furthermore, suppose H is a closed subgroupoid of G with Haar system λ_H. Let $X = s^{-1}(H^{(0)})$, G^H be the imprimitivity groupoid, and σ be the action of G^H on ρ^*A. Then $\mathcal{Z}_0 = \Gamma_c(X, s^*\mathcal{A})$ becomes a pre-$\rho^*A \rtimes_\sigma G^H - A(H^{(0)}) \rtimes_{\alpha|_H} H$-imprimitivity bimodule with respect to the following actions for $f \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$, $g \in \Gamma_c(H, r^*\mathcal{A})$, and $z, w \in \mathcal{Z}_0$:*

$$f \cdot z(\gamma) = \int_G \alpha_\gamma^{-1}(f([\gamma,\eta]))z(\eta)d\lambda_{s(\gamma)}(\eta) \tag{6.3}$$

$$z \cdot g(\gamma) = \int_H \alpha_\eta(z(\gamma\eta)g(\eta^{-1}))d\lambda_H^{s(\gamma)}(\eta) \tag{6.4}$$

$$\langle\langle z, w\rangle\rangle_{A\rtimes H}(\eta) = \int_G z(\xi\eta^{-1})^*\alpha_\eta(w(\xi))d\lambda_{s(\eta)}(\xi) \tag{6.5}$$

$$ {}_{\rho^*A\rtimes G^H}\langle\langle z,w\rangle\rangle([\gamma,\eta]) = \int_H \alpha_{\gamma\xi}(z(\gamma\xi)w(\eta\xi)^*)d\lambda_H^{s(\gamma)}(\xi) \tag{6.6} $$

The completion $\mathcal{Z}_H^G$ of $\mathcal{Z}_0$ is a $\rho^*A \rtimes_\sigma G^H - A(H^{(0)}) \rtimes_{\alpha|_H} H$-imprimitivity bimodule and $\rho^*A \rtimes_\sigma G^H$ and $A(H^{(0)}) \rtimes_\alpha H$ are Morita equivalent.

Proof. Let A, G, α, and H be as above and suppose $\mathcal{A}$ is the upper-semicontinuous bundle associated to A. Let $X = s^{-1}(H^{(0)})$ be the canonical (G^H, H)-equivalence, let s_X be the restriction of the source map to X, and let $r_X : X \to X/H$ be the quotient map. Consider the pull back $\mathfrak{E} = s_X^*(\mathcal{A})$. This is clearly an upper semicontinuous bundle and we define $\mathcal{Z}_0 = \Gamma_c(X, s^*\mathcal{A})$. We will construct an equivalence from $\mathfrak{E}$. Observe that given $\gamma \in X$ we have $\mathfrak{E}_\gamma = \{\gamma\} \times A(s(\gamma))$, $A(s_X(\gamma)) = A(s(\gamma))$ and $\rho^*A(r_X(\gamma)) = \rho^*A(\gamma \cdot H) = A(r(\gamma))$. Thus we can equip $\mathfrak{E}_\gamma$ with the $A(r(\gamma)) - A(s(\gamma))$-imprimitivity bimodule structure coming from the isomorphism α_γ [RW98, Example 3.14]. Specifically, given $a, b \in A(s(\gamma))$ and $c \in A(r(\gamma))$ we have

$$ \begin{aligned} c\cdot(\gamma,a) &= (\gamma, \alpha_\gamma^{-1}(c)a), & (\gamma,a)\cdot b &= (\gamma, ab), \\ {}_{A(r(\gamma))}\langle(\gamma,a),(\gamma,b)\rangle &= \alpha_\gamma(ab^*), & \langle(\gamma,a),(\gamma,b)\rangle_{A(s(\gamma))} &= a^*b. \end{aligned} $$

Next, let $p_{\mathfrak{E}} : \mathfrak{E} \to X$ be the bundle map and define $r_{\mathfrak{E}} : \mathfrak{E} \to X/H$ so that $r_{\mathfrak{E}}(\gamma,a) = \gamma\cdot H$ and $s_{\mathfrak{E}} : \mathfrak{E} \to H^{(0)}$ so that $s(\gamma,a) = s(\gamma)$. These maps are clearly continuous. Furthermore $r_{\mathfrak{E}}$ is just the the composition of $p_{\mathfrak{E}}$ and r_X so that $r_{\mathfrak{E}}$ is open. Similarly $s_{\mathfrak{E}} = s_X \circ p_{\mathfrak{E}}$ so $s_{\mathfrak{E}}$ is open as well. (Recall that s_X is open because X is a saturated closed set in G.) Now we define actions of G^H and H on $\mathfrak{E}$ for $[\eta,\zeta] \in G^H$, $\xi \in H$, and $(\gamma,a) \in \mathfrak{E}$ by

$$ (\gamma,a)\cdot\xi := (\gamma\xi, \alpha_\xi^{-1}(a)) \tag{6.7} $$
$$ [\eta,\zeta]\cdot(\gamma,a) := ([\eta,\zeta]\cdot\gamma, a) = (\eta\zeta^{-1}\gamma, a). \tag{6.8} $$

The second equality in (6.8) follows from the fact that $[\eta,\zeta]\cdot\gamma = \eta\delta$ where δ is the unique element of H such that $\gamma = \zeta\delta$. Notice that (6.8) is well defined because $a \in A(s(\gamma)) = A(s(\eta\zeta^{-1}\gamma))$. Furthermore it is easy to show that the action of G^H on $\mathfrak{E}$ is continuous and respects the groupoid structure. Now consider (6.7). It is also easy to show that this defines an action of H on $\mathfrak{E}$ which is continuous because α is continuous. Thus it follows that $\mathfrak{E}$ is a strong left G^H-space and a strong right H-space. Finally, we recall that the actions of G^H and H on X commute so that

$$ \begin{aligned} ([\eta,\zeta]\cdot(\gamma,a))\cdot\xi &= (([\eta,\zeta]\cdot\gamma)\cdot\xi, \alpha_\xi^{-1}(a)) \\ &= ([\eta,\zeta]\cdot(\gamma\cdot\xi), \alpha_\xi^{-1}(a)) = [\eta,\zeta]\cdot((\gamma,a)\cdot\xi). \end{aligned} $$

Hence the actions on $\mathfrak{E}$ commute.

At this point we need to verify the equivalence conditions on $\mathfrak{Z}$. The continuity condition follows in a straightforward manner from the fact that the operations on $\mathcal{A}$ are continuous and the fact that α is a continuous action. Next we need to show that $p_{\mathfrak{Z}}$ is equivariant. However, it is clear from (6.7) and (6.8) that this is the case. The third condition to verify is compatibility. Suppose $[\eta,\zeta] \in G^H$, $\xi \in H$, $(\gamma,a),(\gamma,b) \in \mathfrak{Z}$, $c \in A(s(\gamma))$, and $d \in A(r(\gamma))$. Then we compute

$$\begin{aligned}
{}_{A(r(\eta))}\langle [\eta,\zeta]\cdot(\gamma,a), [\eta,\zeta]\cdot(\gamma,b)\rangle &= {}_{A(r(\eta))}\langle (\eta\zeta^{-1}\gamma,a),(\eta\zeta^{-1}\gamma,b)\rangle \\
&= \alpha_{\eta\zeta^{-1}\gamma}(ab^*) = \alpha_{\eta\zeta^{-1}}(\alpha_\gamma(ab^*)) \\
&= \sigma_{[\eta,\zeta]}({}_{A(r(\gamma))}\langle(\gamma,a),(\gamma,b)\rangle), \\
\langle (\gamma,a)\cdot\xi, (\gamma,b)\cdot\xi\rangle_{A(s(\xi))} &= \langle (\gamma\xi,\alpha_\xi^{-1}(a)),(\gamma\xi,\alpha_\xi^{-1}(b))\rangle_{A(s(\xi))} = \alpha_\xi^{-1}(a^*b) \\
&= \alpha_\xi^{-1}(\langle(\gamma,a),(\gamma,b)\rangle_{A(s(\gamma))}), \\
[\eta,\zeta]\cdot(d\cdot(\gamma,a)) &= [\eta,\zeta]\cdot(\gamma,\alpha_\gamma^{-1}(d)a) = (\eta\zeta^{-1}\gamma,\alpha_\gamma^{-1}(d)a) \\
&= (\eta\zeta^{-1}\gamma,\alpha_{\gamma^{-1}\zeta\eta^{-1}}(\alpha_{\eta\zeta^{-1}}(d))a) \\
&= (\eta\zeta^{-1}\gamma,\alpha^{-1}_{\eta\zeta^{-1}\gamma}(\sigma_{[\eta,\zeta]}(d))a) \\
&= \sigma_{[\eta,\zeta]}(d)\cdot([\eta,\zeta]\cdot(\gamma,a)), \\
((\gamma,a)\cdot c)\cdot\xi &= (\gamma\xi,\alpha_\xi^{-1}(ac)) = (\gamma\xi,\alpha_\xi^{-1}(a)\alpha_\xi^{-1}(c)) \\
&= ((\gamma,a)\cdot\xi)\cdot\alpha_\xi^{-1}(c).
\end{aligned}$$

This shows that the operations are compatible and all that is left is to verify the invariance condition. Once again, we calculate

$$\begin{aligned}
[\eta,\zeta]\cdot((\gamma,a)\cdot c) &= ([\eta,\zeta]\cdot\gamma, ac) = ([\eta,\zeta]\cdot(\gamma,a))\cdot c, \\
(d\cdot(\gamma,a))\cdot\xi &= (\gamma,\alpha_\gamma^{-1}(d)a)\cdot\xi = (\gamma\xi,\alpha_\xi^{-1}(\alpha_\gamma^{-1}(d)a)) \\
&= (\gamma\xi,\alpha_{\gamma\xi}^{-1}(d)\alpha_\xi^{-1}(a)) = d\cdot(\gamma\xi,\alpha_\xi^{-1}(a)) \\
&= d\cdot((\gamma,a)\cdot\xi).
\end{aligned}$$

At this point we have shown that $\mathfrak{Z}$ is an equivalence between $(A(H^{(0)}),H,\alpha|_H)$ and (ρ^*A,G^H,σ). We can apply Theorem 5.3 to conclude that $\mathcal{Z}_0$ completes to the desired imprimitivity bimodule. What's more we can use (5.1) through (5.4) to compute the bimodule operations. First recall that G^H has a Haar system μ defined by (1.2). Fix $f \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$, $g \in \Gamma_c(H, r^*\mathcal{A})$ and $z,w \in \Gamma_c(X, s^*\mathcal{A})$. Let $z([\eta,\gamma]\cdot\gamma) = z(\eta) = (\eta,a)$ and recall that $[\gamma,\eta]\cdot(\eta,a) = ([\gamma,\eta]\cdot\eta,a) = (\gamma,a)$. Furthermore $f([\gamma,\eta])\cdot(\gamma,a) = (\gamma,\alpha_\gamma^{-1}(f([\gamma,\eta]))a)$. Making the usual identification of a with $z(\eta)$ we then have

$$f\cdot z(\gamma) = \int_{G^H} f([\zeta,\eta])\cdot([\zeta,\eta]\cdot z([\zeta,\eta]^{-1}\cdot\gamma))d\mu^{\gamma\cdot H}([\zeta,\eta])$$

$$= \int_G f([\gamma,\eta]) \cdot ([\gamma,\eta] \cdot z([\eta,\gamma] \cdot \gamma)) d\lambda_{s(\gamma)}(\eta)$$
$$= \int_G \alpha_\gamma^{-1}(f([\gamma,\eta]))z(\eta) d\lambda_{s(\gamma)}(\eta).$$

Similarly we compute

$$\begin{aligned} z \cdot g(\gamma) &= \int_H (z(\gamma \cdot \eta) \cdot \eta^{-1}) \cdot \alpha_\eta(g(\eta^{-1})) d\lambda_H^{s(\gamma)}(\eta) \\ &= \int_H \alpha_\eta(z(\gamma\eta))\alpha_\eta(g(\eta^{-1})) d\lambda_H^{s(\gamma)}(\eta) \\ &= \int_H \alpha_\eta(z(\gamma\eta)g(\eta^{-1})) d\lambda_H^{s(\gamma)}(\eta). \end{aligned}$$

Next, given $\gamma \in H$, in (5.3) we are allowed to choose any $\delta \in X$ such that $s(\delta) = s(\gamma)$. However we may as well choose $\delta = s(\gamma)$ so that

$$\begin{aligned} &\langle\!\langle z,w \rangle\!\rangle_{A \rtimes H}(\gamma) \\ &\quad = \int_{G^H} \langle z([\zeta,\eta]^{-1} \cdot s(\gamma) \cdot \gamma^{-1}), w([\zeta,\eta]^{-1} \cdot s(\gamma)) \cdot \gamma^{-1} \rangle_{A(r(\gamma))} d\mu^{s(\gamma)\cdot H}([\zeta,\eta]) \\ &\quad = \int_G z([\eta, s(\gamma)] \cdot \gamma^{-1})^* \alpha_\gamma(w([\eta, s(\gamma)] \cdot s(\gamma))) d\lambda_{s(\gamma)}(\eta) \\ &\quad = \int_G z(\eta\gamma^{-1})^* \alpha_\gamma(w(\eta)) d\lambda_{s(\gamma)}(\eta). \end{aligned}$$

Finally, given $[\gamma,\eta] \in G^H$ in (5.4) we are allowed to choose any $\delta \in X$ such that $r_X(\delta) = s([\gamma,\eta]) = \eta \cdot H$. Therefore we may as well choose $\delta = \eta$ so that

$$\begin{aligned} {}_{\rho^* A \rtimes_\sigma G^H}\langle\!\langle z,w \rangle\!\rangle([\gamma,\eta]) &= \int_G {}_{A(r(\gamma))}\langle z([\gamma,\eta] \cdot \eta \cdot \xi), [\gamma,\eta] \cdot w(\eta \cdot \xi) \rangle d\lambda^{s(\eta)}(\xi) \\ &= \int_G {}_{A(r(\gamma))}\langle z(\gamma\xi), w(\eta\xi) \rangle d\lambda^{s(\gamma)}(\xi) \\ &= \int_G \alpha_{\gamma\xi}(z(\gamma\xi)^* w(\eta\xi)) d\lambda^{s(\gamma)}(\xi). \qquad \square \end{aligned}$$

Remark 6.4. Suppose G is a transitive groupoid and $u \in G^{(0)}$. It is worth pointing out that Theorem 5.7 becomes a special case of Proposition 6.3 after we identify G with G^{S_u} via Proposition 1.94. We will actually use this fact indirectly in Section 6.2.

Remark 6.5. It is not obvious, but Proposition 6.3 reduces to Green's Imprimitivity Theorem [Wil07, Theorem 4.22] when (A, G, α) is a group dynamical system and H is a subgroup of G. We will sketch this construction without going into detail. First,

observe that $X = G$. It is not difficult to show that the imprimitivity groupoid G^H is isomorphic to the transformation groupoid $G \ltimes G/H$ associated to the left action of G on G/H. As a bundle, A has a single fibre so that $\rho^* A = C_0(G/H, A)$. Proposition 6.1 implies that there is an action σ of $G \ltimes G/H$ on $C_0(G/H, A)$. Now, define $\bar{\sigma} = \mathrm{lt} \otimes \alpha$ and let G act on $C_0(G/H, A)$ via $\bar{\sigma}$. It is straightforward to show, using Example 4.7 as a guide, that $C_0(G/H, A) \rtimes_\sigma (G/H \ltimes G)$ is naturally isomorphic to $C_0(G/H, A) \rtimes_\sigma G$ via the map $\phi(f)(s)([t]) = f(s, [t])$ for $f \in C_c(G, C_c(G/H, A))$. Thus Proposition 6.3 implies that $C_0(G/H, A) \rtimes_\sigma G$ is Morita equivalent to $A \rtimes_\alpha H$, just as in Green's Imprimitivity Theorem. While the imprimitivity algebras from these theorems are not the same, they are related. Since A has a single fibre, $\mathcal{Z}_0 = C_c(G, A)$. Let $\mathcal{X}_0 = C_c(G, A)$ be the pre-imprimitivity bundle coming from Green's Imprimitivity Theorem and define $\psi : \mathcal{Z}_0 \to \mathcal{X}_0$ such that $\psi(f)(s) = \alpha_s(f(s))$. It is not difficult, but requires some lengthy computations, to show that ψ is a bijection which preserves all of the operations defined in Proposition 6.3 and [Wil07, Theorem 4.22].

The whole purpose of building the imprimitivity bimodule $\mathcal{Z}_H^G$ was so that we can mimic [Wil07, Section 5.1] and [Rie74] to create an induction process for representations.

Remark 6.6. It is assumed that the reader is familiar with the material from [RW98, Section 2.4]. We will be making use of the induced representation construction described there. In particular, given a separable groupoid dynamical system (A, G, α), a closed subgroupoid H with a Haar system, and a representation π of $A \rtimes_\alpha H$ on $\mathcal{H}$ we will define the Hilbert space $\mathcal{Z}_H^G \otimes_{A \rtimes H} \mathcal{H}$ to be the completion of the vector space tensor product $\mathcal{Z}_H^G \odot \mathcal{H}$ with respect to the inner product characterized by

$$(z \otimes h, w \otimes k) := (\pi(\langle\langle w, z \rangle\rangle_{A \rtimes H})h, k). \tag{6.9}$$

In any case, our ultimate goal will be to prove the following

Theorem 6.7. *Suppose (A, G, α) is a separable groupoid dynamical system and that H is a closed subgroupoid of G with a Haar system. Then given a representation π of $A(H^{(0)}) \rtimes_\alpha H$ on $\mathcal{H}$ we may form the induced representation $\mathrm{Ind}_H^G \pi$ of $A \rtimes_\alpha G$ on $\mathcal{Z}_H^G \otimes_{A \rtimes H} \mathcal{H}$ which is defined for $f \in \Gamma_c(G, r^*\mathcal{A})$, $z \in \mathcal{Z}_0$ and $h \in \mathcal{H}$ by*

$$\mathrm{Ind}_H^G \pi(f)(z \otimes h) = f \cdot z \otimes h$$

where

$$f \cdot z(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta)) z(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta). \tag{6.10}$$

We start by proving that we can let $A \rtimes G$ act nondegenerately as adjointable linear operators on $\mathcal{Z}_G^H$. Actually, considering the remarks at the end of [RW98, Section 3.3], this gets us most of the way there.

Proposition 6.8. *Let (A, G, α) be a separable groupoid dynamical system, H a closed subgroupoid of G with a Haar system and $\mathcal{Z}_H^G$ be the associated imprimitivity bimodule. Let λ be a Haar system for G, $X = s^{-1}(H^{(0)})$, and $\mathcal{Z}_0 = \Gamma_c(X, s^*\mathcal{A})$. Then there is a nondegenerate homomorphism $\phi : A \rtimes_\alpha G \to \mathcal{L}(\mathcal{Z}_H^G)$ such that for $f \in \Gamma_c(G, r^*\mathcal{A})$ and $z \in \mathcal{Z}_0$*

$$\phi(f)z(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta))z(\eta^{-1}\gamma)d\lambda^{r(\gamma)}(\eta). \tag{6.11}$$

Proof. We will construct ϕ by showing that $A \rtimes G$ sits inside the multiplier algebra of $\rho^*A \rtimes_\sigma G^H$ where G^H is the imprimitivity groupoid and σ is the associated action of G^H on ρ^*A. Given $f \in \Gamma_c(G, r^*\mathcal{A})$ and $g \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ define

$$M_f(g)([\gamma, \eta]) = \int_G f(\xi)\alpha_\xi(g([\xi^{-1}\gamma, \eta]))d\lambda^{r(\gamma)}(\xi). \tag{6.12}$$

We start by proving that $M_f(g)$ is a continuous compactly supported section. This argument is nearly the same as the argument that convolution is well defined. Consider the function

$$\kappa(\xi, [\gamma, \eta]) = f(\xi)\alpha_\xi(g([\xi^{-1}\gamma, \eta]))$$

on $G * G^H = \{(\xi, [\gamma, \eta]) \in G \times G^H : r(\xi) = r(\gamma)\}$. Suppose $\xi_i \to \xi$ in G, $[\gamma_i, \eta_i] \to [\gamma, \eta]$ in G^H, that $r(\xi_i) = r(\gamma_i)$ for all i and $r(\xi) = r(\gamma)$. Then, after passing to a subsequence, choosing new representatives and relabeling, we may assume that $\gamma_i \to \gamma$ and $\eta_i \to \eta$. However it follows immediately that $[\xi_i^{-1}\gamma_i, \eta_i] \to [\xi^{-1}\gamma, \eta]$. From here, one just observes that α is continuous to conclude that κ is a continuous function. Furthermore, suppose the sequence $(\xi_i, [\gamma_i, \eta_i])$ is in $\operatorname{supp}\kappa$. Then we must have $\xi_i \in \operatorname{supp} f$ and $[\xi_i^{-1}\gamma_i, \eta_i] \in \operatorname{supp} g$ for all i. Since both f and g are compactly supported we can find subsequences which converge. However, after passing to another subsequence and choosing new representatives γ_i and η_i we can assume that there exists $\xi, \gamma, \eta \in G$ such that $\xi_i \to \xi$, $\xi_i^{-1}\gamma_i \to \gamma$ and $\eta_i \to \eta$. However it is now clear that $(\xi_i, [\gamma_i, \eta_i])$ has a subsequence which converges to $(\xi, [\gamma\xi, \eta])$. Thus κ is a compactly supported continuous function so that $\kappa \in \Gamma_c(G * G^H, \bar{r}^*\mathcal{A})$, where in this case $\bar{r}(\xi, [\gamma, \eta]) = r(\xi)$.

Now, given $\kappa \in \Gamma_c(G * G^H, \bar{r}^*\mathcal{A})$ we wish to show that the function

$$[\gamma, \eta] \mapsto \int_G \kappa(\xi, [\gamma, \eta])d\lambda^{r(\gamma)}(\xi) \tag{6.13}$$

is continuous and compactly supported. As in Lemma 3.55, if $\kappa_i \to \kappa$ with respect to the inductive limit topology and (6.13) is continuous for each κ_i then it is continuous for κ as well. Since sums of elementary tensors are dense in $\Gamma_c(G * G^H, \bar{r}^*\mathcal{A})$, and since sums of continuous functions are continuous, we may as well assume that $\kappa = h \otimes a$ for $h \in C_c(G * G^H)$ and $a \in A$. It is not difficult to see that $G * G^H$ is closed in $G \times G^H$. As a result we can extend h using Lemma 1.71 to a compactly supported

function on $G \times G^H$. Since sums of functions of the form $k \otimes l(\xi, [\gamma, \eta]) = k(\xi)l([\gamma, \eta])$ are dense in $C_c(G \times G^H)$ we can, as above, assume that $h = k \otimes l$ for $k \in C_c(G)$ and $l \in C_c(G^H)$. However in this case

$$\int_G (k \otimes l) \otimes a(\xi, [\gamma, \eta]) d\lambda^{r(\gamma)}(\xi) = l([\gamma, \eta])a(r(\gamma)) \int_G k(\xi) d\lambda^{r(\gamma)}(\xi)$$

and thus for $\kappa = k \otimes l \otimes a$ (6.13) is clearly a continuous and compactly supported function on G^H. It now follows that $M_f g \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$.

Next, suppose that the sequence $f_i \to f$ with respect to the inductive limit topology in $\Gamma_c(G, r^*\mathcal{A})$ and $g_i \to g$ with respect to the inductive limit topology in $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$. Let K be a compact set in G eventually containing $\operatorname{supp} f_i$ and L a compact set in G^H eventually containing $\operatorname{supp} g_i$. We compute

$$\begin{aligned}
&\|M_{f_i} g_i([\gamma, \eta]) - M_f g([\gamma, \eta])\| \\
&\leq \int_G \|f_i(\xi)\alpha_\xi(g_i([\xi^{-1}\gamma, \eta])) - f(\xi)\alpha_\xi(g([\xi^{-1}\gamma, \eta]))\| d\lambda^{r(\gamma)}(\xi) \\
&\leq \int_G \|f_i(\xi) - f(\xi)\| \|g_i([\xi^{-1}\gamma, \eta])\| + \|f(\xi)\| \|g_i([\xi^{-1}\gamma, \eta]) - g([\xi^{-1}\gamma, \eta])\| d\lambda^{r(\gamma)}(\xi) \\
&\leq (\|f_i - f\|_\infty \|g_i\|_\infty + \|f\|_\infty \|g_i - g\|_\infty)\lambda^{r(\gamma)}(K)
\end{aligned}$$

Since $\{\|g_i\|_\infty\}$ and $\{\lambda^u(K)\}$ are bounded this shows that $M_{f_i} g_i \to M_f g$ uniformly. Furthermore it is straightforward to show that $\{[\gamma\xi^{-1}, \eta] \in G^H : \xi \in K, [\gamma, \eta] \in L\}$ is a compact set which eventually contains $\operatorname{supp} M_{f_i} g_i$. Thus M is jointly continuous with respect to the inductive limit topology.

In order to prove that M_f defines a multiplier we need to show that it extends to an adjointable linear operator when we view $\rho^* A \rtimes_\sigma G^H$ as a right $\rho^* A \rtimes_\sigma G^H$-module in the usual fashion. First, it is clear that M_f is linear. Next we show that M_f preserves the left action of $\rho^* A \rtimes_\sigma G^H$ on $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$. Let μ be the Haar system on G^H from Proposition 1.95. Using the left invariance of Haar measure we compute for $f \in \Gamma_c(G, r^*\mathcal{A})$ and $g, h \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$

$$\begin{aligned}
&M_f(g * h)([\gamma, \eta]) \\
&\qquad = \int_G \int_{G^H} f(\xi)\alpha_\xi(g([\delta, \zeta])\sigma_{[\delta,\zeta]}(h([\delta, \zeta]^{-1}[\xi^{-1}\gamma, \eta]))) d\mu^{\xi^{-1}\gamma \cdot H}([\delta, \zeta]) d\lambda^{r(\gamma)}(\xi) \\
&\qquad = \int_G \int_G f(\xi)\alpha_\xi(g([\xi^{-1}\gamma, \zeta])\alpha_{\xi^{-1}\gamma\zeta^{-1}}(h([\zeta, \eta]))) d\lambda_{s(\gamma)}(\zeta) d\lambda^{r(\gamma)}(\xi) \\
&\qquad = \int_G \int_G f(\xi)\alpha_\xi(g([\xi^{-1}\gamma, \zeta]))\alpha_{\gamma\zeta^{-1}}(h([\zeta, \eta])) d\lambda^{r(\gamma)}(\xi) d\lambda_{s(\gamma)}(\zeta)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{G^H} M_f g([\delta,\zeta])\sigma_{[\delta,\zeta]}(h([\delta,\zeta]^{-1}[\gamma,\eta]))d\mu^{\gamma\cdot H}([\delta,\zeta]) \\
&= (M_f g) * h([\gamma,\eta]).
\end{aligned}
$$

Next, we show that M is adjointable on $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ with adjoint M_{f^*} by computing

$$
\begin{aligned}
(M_f g)^* * h([\gamma,\eta]) &= \int_{G^H} (M_f g)^*([\zeta,\xi])\sigma_{[\zeta,\xi]}(h([\zeta,\xi]^{-1}[\gamma,\eta]))d\mu^{\gamma\cdot H}([\zeta,\xi]) \\
&= \int_G \sigma_{[\gamma,\xi]}((M_f g([\xi,\gamma]))^* h([\xi,\eta]))d\lambda_{s(\gamma)}(\xi) \\
&= \int_G\int_G \alpha_{\gamma\xi^{-1}}(\alpha_\zeta(g([\zeta^{-1}\xi,\gamma]))^* f(\zeta)^* h([\xi,\eta]))d\lambda^{r(\xi)}(\zeta)d\lambda_{s(\gamma)}(\xi) \\
&= \int_G\int_G \alpha_{\gamma\xi\zeta}(g([\zeta^{-1}\xi^{-1},\gamma])^*)\alpha_{\gamma\xi}(f(\zeta)^* h([\xi^{-1},\eta]))d\lambda^{s(\xi)}(\zeta)d\lambda^{s(\gamma)}(\xi) \\
&= \int_G\int_G \alpha_{\gamma\zeta}(g([\zeta^{-1},\gamma])^*)\alpha_{\gamma\xi}(f(\xi^{-1}\zeta)^* h([\xi^{-1},\eta]))d\lambda^{s(\gamma)}(\zeta)d\lambda^{s(\gamma)}(\xi) \\
&= \int_G\int_G \alpha_{\gamma\zeta}(g([\zeta^{-1},\gamma])^* \alpha_{\zeta^{-1}\xi}(f(\xi^{-1}\zeta)^* h([\xi^{-1},\eta])))d\lambda^{s(\gamma)}(\xi)d\lambda^{s(\gamma)}(\zeta) \\
&= \int_G\int_G \alpha_{\gamma\zeta}(g([\zeta^{-1},\gamma])^* \alpha_\xi(f(\xi^{-1})^*)\alpha_\xi(h([\xi^{-1}\zeta^{-1},\eta])))d\lambda^{s(\zeta)}(\xi)d\lambda^{s(\gamma)}(\zeta) \\
&= \int_G\int_G \alpha_{\gamma\zeta^{-1}}(g([\zeta,\gamma])^* f^*(\xi)\alpha_\xi(h([\xi^{-1}\zeta,\eta])))d\lambda^{r(\zeta)}(\xi)d\lambda_{s(\gamma)}(\zeta) \\
&= \int_G \sigma_{[\gamma,\zeta]}(g([\zeta,\gamma])^* M_{f^*}(h)([\zeta,\eta]))d\lambda_{s(\gamma)}(\zeta) \\
&= \int_{G^H} \sigma_{[\xi,\zeta]}(g([\xi,\zeta]^{-1})^* M_{f^*}(h)([\xi,\zeta]^{-1}[\gamma,\eta]))d\mu^{\gamma\cdot H}([\xi,\zeta]) \\
&= g^* * (M_{f^*}h)([\gamma,\eta]).
\end{aligned}
$$

Finally, we prove that M preserves convolution on $\Gamma_c(G, r^*\mathcal{A})$ by computing for $f, g \in \Gamma_c(G, r^*\mathcal{A})$ and $h \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ that

$$
\begin{aligned}
M_{f*g}h([\gamma,\eta]) &= \int_G\int_G f(\delta)\alpha_\delta(g(\delta^{-1}\xi))\alpha_\xi(h([\xi^{-1}\gamma,\eta]))d\lambda^{r(\gamma)}(\delta)d\lambda^{r(\gamma)}(\xi) \\
&= \int_G\int_G f(\delta)\alpha_\delta(g(\delta^{-1}\xi)\alpha_{\delta^{-1}\xi}(h([\xi^{-1}\gamma,\eta]))d\lambda^{r(\gamma)}(\xi)d\lambda^{r(\gamma)}(\delta) \\
&= \int_G\int_G f(\delta)\alpha_\delta(g(\xi)\alpha_\xi(h([\xi^{-1}\delta^{-1}\gamma,\eta])))d\lambda^{s(\delta)}(\xi)d\lambda^{r(\gamma)}(\delta)
\end{aligned}
$$

$$= \int_G f(\delta)\alpha_\delta(M_g h([\delta^{-1}\gamma,\eta]))d\lambda^{r(\gamma)}(\delta)$$
$$= M_f M_g h([\gamma,\eta]).$$

Next we show that M is nondegenerate. Specifically we will show that elements of the form $M_f(g)$ are dense in $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ with respect to the inductive limit topology. Let a_l be an approximate identity for A and let $e_{(K,U,l,\epsilon)}$ be the approximate identity coming from Lemma 5.10. We will use κ to denote a generic 4-tuple (K,U,l,ϵ). We would like to show that given $g \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ we have $M_{e_\kappa} g \to g$ with respect to the inductive limit topology. Fix $\epsilon_1 > 0$ and let $L = \operatorname{supp} g$. We make the following claim.

Claim. There exists a conditionally compact neighborhood U_1 such that $\xi \in U_1$ implies

$$\|\alpha_\xi(g([\xi^{-1}\gamma,\eta])) - g([\gamma,\eta])\| < \epsilon_1 \tag{6.14}$$

for all $[\gamma,\eta] \in G^H$ such that $r(\gamma) = r(\xi)$.

Proof of Claim. Suppose the claim does not hold. Fix some conditionally compact neighborhood W. Then for any conditionally compact neighborhood $U \subset W$ there exists $\xi_U \in U$ and $[\gamma_U, \eta_U] \in G^H$ such that

$$\|\alpha_{\xi_U}(g([\xi_U^{-1}\gamma_U,\eta_U])) - g([\gamma_U,\eta_U])\| \geq \epsilon_1.$$

However, for this to hold one of the terms must be nonzero so that we must have, recalling that W is a neighborhood of $G^{(0)}$ which contains U,

$$[\gamma_U,\eta_U] \in \widetilde{L} = \{[\xi\gamma,\eta] : \xi \in W, [\gamma,\eta] \in L \text{ and } s(\xi) = r(\gamma)\}. \tag{6.15}$$

Suppose $\{[\xi_i^{-1}\gamma_i,\eta_i]\}$ is contained in $\widetilde{L}$. Then, by passing to a subsequence, relabeling and choosing new representatives we can use the fact that L is compact to find $\gamma,\eta \in G$ such that $\gamma_i \to \gamma$ and $\eta_i \to \eta$. Since ρ is continuous it follows that $K_1 = \rho(r(L))$ is compact and by assumption contains $\{s(\xi_i)\}$. Since W is conditionally compact the set $W \cap s^{-1}(K_1)$ is compact and also contains $\{\xi_i\}$. Now we can pass to another subnet and find ξ such that $\xi_i \to \xi$. It follows immediately that $\widetilde{L}$ is compact. Thus, ordering $\{[\gamma_U,\eta_U]\}$ by decreasing U, we can pass to a subnet (twice actually), relabel and find new representatives such that there exists $\gamma,\eta \in G$ with $\gamma_U \to \gamma$ and $\eta_U \to \eta$. Next, observe that we have $r(\xi_U) \in \rho(r(\widetilde{L}))$ for all U so that $\{\xi_U\}$ is contained in the compact set $U \cap r^{-1}(\rho(r(\widetilde{L})))$. Therefore we can pass to yet another subnet and find ξ such that $\xi_U \to \xi$. However, by construction, $\xi \in U$ for any conditionally compact neighborhood of $G^{(0)}$. It follows that $\xi \in G^{(0)}$. Hence

$$\alpha_{\xi_U}(g([\xi_U^{-1}\gamma_U,\eta_U])) \to g([\gamma,\eta]).$$

However, it follows that eventually

$$\|\alpha_{\xi_U}(g([\xi_U^{-1}\gamma_U, \eta_U])) - g([\gamma_U, \eta_U])\| < \epsilon_1$$

which is a contradiction. $\square$

Unfortunately we need another claim before we can tackle nondegeneracy.

Claim. There exists l_1 such that $l \geq l_1$ implies

$$\|a_l(r(\gamma))g([\gamma, \eta]) - g([\gamma, \eta])\| < \epsilon_1 \quad \text{for all } [\gamma, \eta] \in G^H. \tag{6.16}$$

Proof of Claim. It clearly suffices to verify this identity on L. Since a_l factors to an approximate identity on each fibre we have $a_l(r(\gamma))g([\gamma, \eta]) \to g([\gamma, \eta])$ for each $[\gamma, \eta] \in G^H$. We use the fact that the norm is upper-semicontinuous to choose for each $[\gamma, \eta] \in L$ some neighborhood $O_{[\gamma,\eta]}$ of $[\gamma, \eta]$ and some $b_{[\gamma,\eta]} \in \{a_l\}$ such that

$$\|b_{[\gamma,\eta]}(r(\xi))g([\xi, \zeta]) - g([\xi, \zeta])\| < \frac{\epsilon_1}{3}$$

for all $[\xi, \zeta] \in O_{[\gamma,\eta]}$. Since L is compact we can find some finite subcover $\{O_i\}$. Let $\phi_i \in C_c(G^H)$ be a partition of unity with respect to $\{O_i\}_{i=1}^N$ so that $\operatorname{supp}\phi_i \subset O_i$ and $\sum \phi_i([\gamma, \eta]) = 1$ if $[\gamma, \eta] \in L$. Define $h \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ by

$$h = \sum_{i=1}^{N} \phi_i \otimes b_{[\gamma_i,\eta_i]}.$$

Then by construction, for all $[\xi, \zeta] \in L$ we have

$$\begin{aligned}\|h([\xi, \zeta])g([\xi, \zeta]) - g([\xi, \zeta])\| &\leq \sum_{i=1}^{\infty} \phi_i([\xi, \zeta])\|b_{[\gamma_i,\eta_i]}(r(\xi))g([\xi, \zeta]) - g([\xi, \zeta])\| \\ &< \frac{\epsilon_1}{3}.\end{aligned} \tag{6.17}$$

Moving on, we can find l_1 such that if $l \geq l_1$ then

$$\|a_l b_{[\gamma_i,\eta_i]} - b_{[\gamma_i,\eta_i]}\| < \frac{\epsilon_1}{3\|g\|_\infty}$$

for all i. Using the fact that passing to fibres is norm contractive this implies that for $[\xi, \zeta] \in L$ we have

$$\|a_l(r(\xi))h([\xi, \zeta]) - h([\xi, \zeta])\| \leq \sum_{i=1}^{N} \phi_i([\zeta, \xi])\|a_l(r(\xi))b_{[\gamma_i,\eta_i]}(r(\xi)) - b_{[\gamma_i,\eta_i]}(r(\xi))\|$$

$$< \frac{\epsilon_1}{3\|g\|_\infty}. \tag{6.18}$$

Therefore, using (6.17) and (6.18) and the fact that $\|a_l\| \leq 1$ we compute for $l \geq l_1$ and $[\xi, \zeta] \in L$

$$\begin{aligned}\|a_l(r(\xi))g([\xi,\zeta]) - g([\xi,\zeta])\| \leq &\|a_l(r(\xi))(g([\xi,\zeta]) - h([\xi,\zeta])g([\xi,\zeta]))\| \\ &+ \|(a_l(r(\xi))h([\xi,\zeta]) - h([\xi,\zeta]))g([\xi,\zeta])\| \\ &+ \|h([\xi,\zeta])g([\xi,\zeta]) - g([\xi,\zeta])\| \\ <&\|a_l(r(\xi))\|\frac{\epsilon_1}{3} + \|g\|_\infty \frac{\epsilon_1}{3\|g\|_\infty} + \frac{\epsilon_1}{3} \leq \epsilon_1. \qquad \square\end{aligned}$$

Now suppose we are given $\epsilon_0 > 0$ and let $\epsilon_1 = \epsilon_0/(5 + \|g\|_\infty)$. Choose U_1 and l_1 for ϵ_1 as above. Then given $e = e_{(K,U,l,\epsilon)}$ with $K_1 = \rho(r(L)) \subset K$, $U \subset U_1$, $l_1 \leq 1$ and $\epsilon < \epsilon_1$ we compute for $[\gamma, \eta] \in G^H$

$$\begin{aligned}\|M_e g([\gamma,\eta]) - g([\gamma,\eta])\| \leq &\left\| \int_G e(\xi)(\alpha_\xi(g([\xi^{-1}\gamma,\eta])) - g([\gamma,\eta]))d\lambda^{r(\gamma)}(\xi) \right\| \\ &+ \left\| \left(\int_G e(\xi) d\lambda^{r(\gamma)}(\xi) - a_l(r(\gamma)) \right) g([\gamma,\eta]) \right\| \\ &+ \|a_l(r(\gamma))g([\gamma,\eta]) - g([\gamma,\eta])\| \\ < &\int_U \|e(\xi)\| \|\alpha_\xi(g([\xi^{-1}\gamma,\eta])) - g([\gamma,\eta])\| d\lambda^{r(\gamma)}(\xi) \\ &+ \epsilon \|g([\gamma,\eta])\| + \epsilon_1 \\ <&4\epsilon_1 + \epsilon_1\|g\|_\infty + \epsilon_1 = \epsilon_0.\end{aligned}$$

Thus $M_{e_\kappa} g \to g$ uniformly. Furthermore, if $\kappa = (K, U, l, \epsilon)$ such that $U \subset U_1$ then, considering the fact that $\operatorname{supp} e_\kappa \subset U$ we have $M_{e_\kappa} g([\gamma, \eta]) \neq 0$ only if

$$[\gamma,\eta] \in \widetilde{L} = \{[\xi\gamma,\eta] : \xi \in U_1, [\gamma,\eta] \in L, \text{ and } s(\xi) = r(\gamma)\}$$

However this set has the same form as (6.15) and we proved that $\widetilde{L}$ was compact there. Thus we eventually have $\operatorname{supp} M_{e_\kappa} g \subset \widetilde{L}$ so that $M_{e_\kappa} g \to g$ with respect to the inductive limit topology. This of course implies that elements of the form $M_f g$ are dense in $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ with respect to the inductive limit topology.

This nondegeneracy argument was a bear but it will be crucial in what follows. We would like to show that M_f extends to an adjointable operator on $\rho^* A \rtimes_\sigma G^H$ and that $\|M_f\| \leq \|f\|$ so that M extends to $A \rtimes G$. Well, suppose τ is a state on $\rho^* A \rtimes_\sigma G^H$ and, following the usual GNS construction, define a pre-inner product on

$\rho^* A \rtimes_\sigma G^H$ via
$$(g,h)_\tau := \tau(g^* * h).$$
Let $\mathcal{H}_\tau$ denote the resulting Hilbert space and $\mathcal{H}_0$ the image of $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ in $\mathcal{H}_\tau$. Observe that $\mathcal{H}_0$ is a dense subspace. Now given $f \in \Gamma_c(G, r^*\mathcal{A})$ we would like to define an operator $\pi(f)$ on $\mathcal{H}_0$ by
$$\pi(f)g = M_f g.$$
Of course, we need to see that this factors correctly. Suppose $g \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ is such that $(g,h)_\tau = 0$ for all $h \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$. Then
$$(\pi(f)g,h)_\tau = \tau((M_f g)^* * h) = \tau(g^* * (M_{f^*}h)) = (g, \pi(f^*)h)_\tau = 0$$
for all $h \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ so that $\pi(f)g = 0$. Thus π is well defined and obviously defines a linear operator on $\mathcal{H}_0$. Furthermore, it follows from the fact that M is linear in f and that $M_{f*g} = M_f M_g$ that π is a homomorphism from $\Gamma_c(G, r^*\mathcal{A})$ into the algebra of linear operators on $\mathcal{H}_0$. We will now show that we can apply Theorem 3.119. Since elements of the form $M_f(g)$ are dense in the inductive limit topology it is clear that elements of the form $\pi(f)h$ are dense in $\mathcal{H}_\tau$. Furthermore, we have shown that $M_f g$ is jointly continuous with respect to the inductive limit topology. Therefore, if we fix $g, h \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ and let $f_i \to f$ with respect to the inductive limit topology we must have
$$(\pi(f_i)g,h)_\tau = \tau((M_{f_i}g)^* * h) \to \tau((M_f g)^* * h) = (\pi(f)g,h)_\tau.$$
Finally it is clear from the fact that M_f is adjointable on $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ with adjoint M_{f^*} that
$$(\pi(f)g,h)_\tau = \tau((M_f g)^* * h) = \tau(g^* * (M_{f^*}h)) = (g, \pi(f^*)h)_\tau.$$
Thus we may apply Theorem 3.119 and conclude that π extends to representation of $A \rtimes G$. In particular this implies that for $f \in \Gamma_c(G, r^*\mathcal{A})$ and $g \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ we have
$$\begin{aligned}\tau((M_f g)^* * (M_f g)) &= \|\pi(f)g\|_\tau^2 \leq \|f\|^2\|g\|_\tau^2 = \|f\|^2\tau(g^* * g)\\ &\leq \|f\|^2\|g^* * g\| = \|f\|^2\|g\|^2.\end{aligned}$$
However, τ is an arbitrary state on $\rho^* A \rtimes_\sigma G^H$ so that by choosing τ such that $\tau((M_f g)^* * (M_f g)) = \|M_f g\|^2$ [RW98, Lemma A.3] we must have
$$\|M_f g\| \leq \|f\|\|g\|. \tag{6.19}$$

Therefore M_f is bounded and as such extends to a linear operator on $\rho^*A \rtimes_\sigma G^H$. However, M_f is $\rho^*A \rtimes_\sigma G^H$-linear and adjointable on a dense subspace so that this must be true in general. It follows that M_f defines a multiplier on $\rho^*A \rtimes_\sigma G^H$. Furthermore it is clear that $f \mapsto M_f$ is linear and we have already verified that it preserves convolution on $\Gamma_c(G, r^*\mathcal{A})$. In addition we calculated that the adjoint of M_f is M_{f^*} so that M preserves involution as well. Since (6.19) implies that $\|M_f\| \leq \|f\|$ it follows that M extends to a $*$-homomorphism from $A \rtimes G$ into $M(\rho^*A \rtimes_\sigma G^H)$.

At this point we are essentially done since the space of multipliers on $\rho^*A \rtimes_\sigma G^H$ can be identified with $\mathcal{L}(\mathcal{Z}_H^G)$. Specifically let $\mathcal{Z}_H^G$ be the imprimitivity bimodule associated to G and H. Recall that since $\mathcal{Z}_H^G$ is a Hilbert $\rho^*A \rtimes_\sigma G^H$-module every element of $\mathcal{Z}$ is of the form $g \cdot z$ for $g \in \rho^*A \rtimes_\sigma G^H$ and $z \in Z$ [RW98, Proposition 2.31]. As such, given $f \in A \rtimes G$ we can define $\phi(f)$ on $\mathcal{Z}$ by setting

$$\phi(f)(g \cdot z) = M_f g \cdot z$$

whenever $g \in \rho^*A \rtimes_\sigma G^H$ and $z \in \mathcal{Z}_H^G$. It is clear that $\phi(f)$ defines linear operator on Z. Next, we compute for $f \in A \rtimes G$, $g, h \in \rho^*A \rtimes_\sigma G^H$ and $z, w \in \mathcal{Z}$ that

$$\begin{aligned}\langle\langle \phi(f) g \cdot z, h \cdot w \rangle\rangle_{A \rtimes H} &= \langle\langle g \cdot z, (M_f g)^* h \cdot w \rangle\rangle_{A \rtimes H} = \langle\langle z, g^* M_{f^*} h \cdot w \rangle\rangle_{A \rtimes H} \\ &= \langle\langle g \cdot z, M_{f^*} h \cdot w \rangle\rangle_{A \rtimes H} = \langle\langle g \cdot z, \phi(f^*) h \cdot w \rangle\rangle_{A \rtimes H}\end{aligned}$$

Thus $\phi(f)$ is adjointable with adjoint $\phi(f^*)$ and it follows that $\phi : A \rtimes G \to \mathcal{L}(\mathcal{Z})$. Furthermore ϕ preserves involution and it is easy to see that ϕ is linear. We now calculate

$$\phi(f * g) h \cdot z = M_{f*g} h \cdot z = M_f M_g h \cdot z = \phi(f)\phi(g) h \cdot z.$$

Thus ϕ is a $*$-homomorphism. Finally, since elements of the form $M_f g$ are dense in $\rho^*A \rtimes_\sigma G^H$, it follows that elements of the form $\phi(f) g \cdot z$ are dense in $\mathcal{Z}$. Hence ϕ is nondegenerate. To finish we calculate for $f \in \Gamma_c(G, r^*\mathcal{A})$, $g \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ and $z \in \mathcal{Z}_0$

$$\begin{aligned}\phi(f) g \cdot z(\gamma) &= \int_G \alpha_\gamma^{-1}(M_f g([\gamma, \eta])) z(\eta) d\lambda_{s(\gamma)}(\eta) \\ &= \int_G \int_G \alpha_\gamma^{-1}(f(\xi) \alpha_\xi(g[\xi^{-1}\gamma, \eta])) z(\eta) d\lambda^{r(\gamma)}(\xi) d\lambda_{s(\gamma)}(\eta) \\ &= \int_G \alpha_\gamma^{-1}(f(\xi)) \int_G \alpha_{\xi^{-1}\gamma}^{-1}(g([\xi^{-1}\gamma, \eta])) z(\eta) d\lambda_{s(\gamma)}(\eta) d\lambda^{r(\gamma)}(\xi) \\ &= \int_G \alpha_\gamma^{-1}(f(\xi)) g \cdot z(\xi^{-1}\gamma) d\lambda^{r(\gamma)}(\xi).\end{aligned}$$

Thus ϕ is given by (6.11) on elements of the form $g \cdot z$ when $g \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ and $z \in \mathcal{Z}_0$. However, it follows from Lemma 5.10 that there is a net in $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ which is an approximate identity in the inductive limit topology with respect to the left action of $\Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ on $\mathcal{Z}_0$. This implies that elements of the form $g \cdot z$ with $g \in \Gamma_c(G^H, r^*(\rho^*\mathcal{A}))$ and $z \in \mathcal{Z}_0$ are dense in $\mathcal{Z}_0$ with respect to the inductive limit topology. Fix $f \in \Gamma_c(G, r^*\mathcal{A})$ and given $z \in \mathcal{Z}_0$ define $\hat{z}(\xi, \gamma) = \alpha_\gamma^{-1}(f(\xi))z(\xi^{-1}\gamma)$ and $\bar{z}(\gamma) = \int_G \hat{z}(\xi, \gamma) d\lambda^{r(\gamma)}(\xi)$. If $z_i \to z$ with respect to the inductive limit topology then it is straightforward to show $\hat{z}_i \to \hat{z}$ with respect to the inductive limit topology and in turn that $\bar{z}_i \to \bar{z}$ with respect to the inductive limit topology. However, it now follows in a straightforward fashion that (6.11) holds in general on $\mathcal{Z}_0$. $\square$

The upshot of all of this is that we can now prove the desired induction theorem.

Proof of Theorem 6.7. Using ϕ from Proposition 6.8 we can let $A \rtimes G$ act nondegenerately as adjointable operators on the Hilbert $A \rtimes_\alpha H$-module $\mathcal{Z}_H^G$. However [RW98, Proposition 2.66] then implies that there is a nondegenerate induced representation $\mathcal{Z} - \operatorname{Ind}(\pi)$ of $A \rtimes G$ acting on elementary tensors in $\mathcal{Z} \otimes_{A \rtimes H} \mathcal{H}$ by

$$\mathcal{Z} - \operatorname{Ind}(\pi)(f)(z \otimes h) = \phi(f)z \otimes h.$$

Thus we can define $\operatorname{Ind}_H^G \pi = \mathcal{Z} - \operatorname{Ind}(\pi)$ and (6.11) shows that $\operatorname{Ind}_H^G \pi$ has the desired action on $\mathcal{Z} \otimes_{A \rtimes H} \mathcal{H}$. $\square$

Example 6.9. Suppose (A, G, α) is a separable dynamical system. Then for any $u \in G^{(0)}$ the stabilizer subgroup S_u is a closed subgroup with a Haar system. Therefore we can induce representations from $A(u) \rtimes S_u$ to $A \rtimes G$.

Example 6.10. Suppose (A, G, α) is a separable dynamical system and π is a representation of A on $\mathcal{H}$. Without loss of generality assume that there exists a Borel Hilbert bundle $G^{(0)} * \mathfrak{H}$ and measure μ so that $\mathcal{H} = L^2(G^{(0)} * \mathfrak{H}, \mu)$ and π decomposes as $\pi = \int_{G^{(0)}}^\oplus \pi_u d\mu(u)$. Consider that $G^{(0)}$ is a closed subgroupoid of G with Haar system given by the δ measures. We can then form the induced representation $\operatorname{Ind}_{G^{(0)}}^G \pi$. Observe that in this situation $A \rtimes G^{(0)}$ is the completion of $\Gamma_c(G^{(0)}, \mathcal{A})$ and that the I-norm on $A \rtimes G^{(0)}$ is just the uniform norm. Since the uniform norm is a C^*-norm, the enveloping algebra is just the uniform norm completion of $\Gamma_c(G^{(0)}, \mathcal{A})$. In other words $A \rtimes G^{(0)} = A$. The right hand operations on $\mathcal{Z}_{G^{(0)}}^G$ then simplify to

$$z \cdot g(\gamma) = z(\gamma)g(s(\gamma))$$
$$\langle\langle z, w \rangle\rangle_A(u) = \int_G z(\eta)^* w(\eta) d\lambda_{s(\gamma)}(\eta).$$

Thus $\mathcal{Z} \otimes_A \mathcal{H}$ is the completion of $\mathcal{Z}_0 = \Gamma_c(G, s^*\mathcal{A})$ with respect to the inner product

$$\begin{aligned}(f \otimes h, g \otimes k) &:= \int_G (\pi(g(\eta)^* f(\eta))h, k) d\lambda_{s(\gamma)}(\eta) \\ &= \int_G (\pi_{s(\eta)}(f(\eta)^* f(\eta))h(s(\eta)), k(s(\eta))) d\nu^{-1}(\eta) \\ &= \int_G (\pi_{s(\eta)}(f(\eta))h(s(\eta)), \pi_{s(\eta)}(g(\eta))h(s(\eta))) d\nu^{-1}(\eta).\end{aligned}$$

where $\nu^{-1} = \int_{G^{(0)}} \lambda_u d\mu(u)$. Furthermore the action of $\operatorname{Ind}\pi$ on $\mathcal{Z} \otimes_A \mathcal{H}$ is determined by

$$\operatorname{Ind}\pi(f)(g \otimes h) = f \cdot g \otimes h$$

where

$$f \cdot g(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta)) g(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta).$$

Now recall that in Example 3.122 we defined the left regular representation L of π to act on $L^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$ via

$$L(f)h(\gamma) = \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\eta))) h(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta).$$

Without going into the details, it is (more or less) straightforward to show that the map $U : \mathcal{Z}_0 \odot \mathcal{H} \to \mathcal{L}^2(s^*(G^{(0)} * \mathfrak{H}), \mu)$ characterized by

$$U(f \otimes h)(\gamma) = \pi_{s(\gamma)}(f(\gamma))h(s(\gamma))$$

extends to a unitary map $U : \mathcal{Z} \otimes_A \mathcal{H} \to L^2(s^*(G^{(0)} * \mathfrak{H}), \nu^{-1})$. Furthermore

$$\begin{aligned}L(f)U(g \otimes h)(\gamma) &= \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\eta))) U(g \otimes h)(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta) \\ &= \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\eta)) g(\eta^{-1}\gamma)) h(s(\gamma)) d\lambda^{r(\gamma)}(\eta) \\ &= \pi_{s(\gamma)}(f \cdot g(\gamma)) h(s(\gamma)) = U(f \cdot g \otimes h)(\gamma) \\ &= U \operatorname{Ind}\pi(f)(g \otimes h)(\gamma).\end{aligned}$$

Thus the left regular representation associated to π is equivalent to the induced representation $\operatorname{Ind}_{G^{(0)}}^G \pi$.

6.2 Stabilizers and T_0 Orbits

As we noted in Example 6.9, given a groupoid crossed product $A \rtimes G$ and $u \in G^{(0)}$ we can induce representations of $A(u) \rtimes S_u$ to $A \rtimes G$. Since $A(u) \rtimes S_u$ is a group crossed product its representation theory is relatively well understood. In this section we will find conditions so that every irreducible representation of $A \rtimes G$ can be obtained in this fashion. In particular we will consider the "nice" groupoids for which the conditions of the Mackey-Glimm dichotomy hold.

Remark 6.11. Recall that G acts on $G^{(0)}$ by left translation. We denote the image of u in $G^{(0)}/G$ by $G \cdot u$. However this notation is also used for the orbit of u in $G^{(0)}$. We will regularly confuse the two and place the burden of deciding which interpretation to use on the reader.

The key to this section will be to reduce to the case where the orbit space is Hausdorff, because in this case we get the following result.

Proposition 6.12. *Suppose (A, G, α) is a separable groupoid dynamical system and $G^{(0)}/G$ is Hausdorff. Then $A \rtimes_\alpha G$ is a $C_0(G^{(0)}/G)$-algebra with the action Φ defined for $\phi \in C_0(G^{(0)}/G)$ and $f \in \Gamma_c(G, r^*\mathcal{A})$ by*

$$\Phi(\phi)f(\gamma) = \phi(G \cdot r(\gamma))f(\gamma).$$

Furthermore, restriction factors to an isomorphism of $A \rtimes G(G \cdot u)$ onto the fibre $A(G \cdot u) \rtimes G|_{G \cdot u}$.

Proof. First recall that $G^{(0)}/G$ is always locally compact so that in this case $G^{(0)}/G$ is a second countable locally compact Hausdorff space. Suppose Φ is defined as above. It is clear that Φ is at least linear in ϕ and f. Furthermore we check for $f, g \in \Gamma_c(G, r^*\mathcal{A})$ that

$$\begin{aligned}(\Phi(\phi)f)^* * g(\gamma) &= \int_G \alpha_\eta((\Phi(\phi)f(\eta^{-1}))^* g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \qquad (6.20)\\ &= \int_G \alpha_\eta(f(\eta^{-1})^* \overline{\phi(G \cdot s(\eta))} g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta)\\ &= \int_G \alpha_\eta(f(\eta^{-1})^* \Phi(\overline{\phi}) g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta)\\ &= f^* * (\Phi(\overline{\phi})g)(\gamma).\end{aligned}$$

This shows that $\Phi(\phi)$ is adjointable on $\Gamma_c(G, r^*\mathcal{A})$. Next we check that

$$\Phi(\phi)(f * g)(\gamma) = \phi(G \cdot r(\gamma)) \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \qquad (6.21)$$

$$
\begin{aligned}
&= \int_G \Phi(\phi) f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \\
&= (\Phi(\phi)f) * g.
\end{aligned}
$$

This shows that $\Phi(\phi)$ is linear with respect to the left action of $\Gamma_c(G, r^*\mathcal{A})$ on itself. Now let $C_0(G^{(0)}/G)^1$ be the unitization of $C_0(G^{(0)}/G)$ and extend Φ to $C_0(G^{(0)}/G)^1$ by setting $\Phi(\phi + \lambda 1)f = \Phi(\phi)f + \lambda f$. It is a simple matter to show that (6.20) and (6.21) extend to $C_0(G^{(0)}/G)^1$. Let $\langle f, g\rangle = f^* * g$ be the usual inner product on $A \rtimes G$ as an $A \rtimes G$-module. We would like to show that $\|\Phi(\phi)f\| \leq \|\phi\|_\infty \|f\|$ for all $f \in \Gamma_c(G, r^*\mathcal{A})$. It will suffice to show that

$$
\|\phi\|_\infty^2 \langle f, f\rangle - \langle \Phi(\phi)f, \Phi(\phi)f\rangle \geq 0
$$

as elements of $A \rtimes G$. However, using the fact that Φ is adjointable on $C_0(G^{(0)}/G)^1$, this amounts to showing

$$
\langle \Phi(\|\phi\|_\infty^2 1 - \overline{\phi}\phi)f, f\rangle \geq 0. \tag{6.22}
$$

All elements of the form $\|\phi\|_\infty^2 1 - \overline{\phi}\phi$ are positive in $C_0(G^{(0)}/G)^1$ so there exists $\xi \in C_0(G^{(0)}/G)^1$ such that $\|\phi\|_\infty^2 1 - \overline{\phi}\phi = \xi^*\xi$. Therefore we have

$$
\langle \Phi(\|\phi\|_\infty^2 1 - \overline{\phi}\phi)f, f\rangle = \langle \Phi(\xi^*\xi)f, f\rangle = \langle \Phi(\xi)f, \Phi(\xi)f\rangle \geq 0.
$$

It follows that $\Phi(\phi)$ is a bounded operator on $\Gamma_c(G, r^*\mathcal{A})$ with norm less than $\|\phi\|_\infty$. Thus $\Phi(\phi)$ extends to an operator on $A \rtimes_\alpha G$. Furthermore (6.20) and (6.21) imply that $\Phi(\phi)$ is linear with respect to the action of $A \rtimes S$ on itself and that $\Phi(\phi)$ is adjointable with adjoint $\Phi(\phi)^* = \Phi(\overline{\phi})$. Hence $\Phi(\phi) \in M(A \rtimes G)$. We have already shown that Φ preserves the involution on $C_0(G^{(0)}/G)$ and the computation

$$
\Phi(\phi)\Phi(\psi)f(\gamma) = \phi(G \cdot r(\gamma))\psi(G \cdot r(\gamma))f(\gamma) = \Phi(\phi\psi)f(\gamma)
$$

shows that it preserves multiplication as well. Thus Φ is a $*$-homomorphism. In order to show that Φ maps into the center it will suffice to show, using Lemma 4.30, that $\Phi(f*g) = f*\Phi(g)$ for all $f, g \in \Gamma_c(G, r^*\mathcal{A})$. However, observe that $r(\eta^{-1}\gamma) = \eta^{-1}\cdot r(\gamma)$ so that

$$
\begin{aligned}
\Phi(\phi)(f * g)(\gamma) &= \phi(G \cdot r(\gamma)) \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \\
&= \int_G f(\eta)\alpha_\eta(\phi(G \cdot r(\eta^{-1}\gamma))g(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \\
&= f * \Phi(\phi)g.
\end{aligned}
$$

Thus $\Phi(f) \in ZM(A \rtimes G)$. The last thing we need to do is to show that the set

$\Phi(C_0(G^{(0)}/G)) \cdot A \rtimes S$ is dense in $A \rtimes S$. However given $f \in \Gamma_c(G, r^*\mathcal{A})$ the image of $r(\operatorname{supp} f)$ in $G^{(0)}/G$, denoted $K = G \cdot r(\operatorname{supp} f)$, is compact. Therefore we can find $\phi \in C_c(G^{(0)}/G)$ which is one on K and zero off some neighborhood of K. It is easy to see that in this case $\Phi(\phi)f = f$ and it follows immediately that Φ is nondegenerate.

Next we identify the fibers of $A \rtimes G$. Fix $u \in G^{(0)}$. Since $G^{(0)}/G$ is Hausdorff, $G \cdot u$ is closed in $G^{(0)}$. Let $O = G^{(0)} \setminus G \cdot u$. It is clear that $G \cdot u$ and O are both G-invariant so that we may apply Theorem 5.22 to conclude that the restriction map ρ factors to an isomorphism of $A \rtimes G/\ker \rho$ with $A(G \cdot u) \rtimes G|_{G \cdot u}$. Now define

$$I_u = \overline{\operatorname{span}}\{\Phi(\phi)f : \phi \in C_0(G^{(0)}/G), f \in \Gamma_c(G, r^*\mathcal{A}), \phi(G \cdot u) = 0\}.$$

Since, by definition, $A \rtimes G(G \cdot u) = A \rtimes G/I_u$ it will suffice to show that $I_u = \ker \rho$. If $\phi(G \cdot u) = 0$ then $\Phi(\phi)f(\gamma) = 0$ for all $\gamma \in G|_{G \cdot u}$ so that we must have $\Phi(\phi)f \in \ker \rho$. It follows that $I_u \subset \ker \rho$. On the other hand we also know from Theorem 5.22 that $\ker \rho = \operatorname{Ex}(O)$ where $\operatorname{Ex}(O)$ is the ideal generated by those functions $f \in \Gamma_c(G, r^*\mathcal{A})$ such that $\operatorname{supp} f \subset G|_O$. Now let $q : G^{(0)} \to G^{(0)}/G$ be the quotient map. Given $f \in \Gamma_c(G, r^*\mathcal{A})$ such that $\operatorname{supp} f \subset G|_O$ we must have $q(r(\operatorname{supp} f))$ disjoint from $G \cdot u$ in $G^{(0)}/G$. Since $q(r(\operatorname{supp} f))$ is compact we can find some $\phi \in C_c(G^{(0)}/G)$ such that ϕ is one on $q(r(\operatorname{supp} f))$ and $\phi(G \cdot u) = 0$. It follows that $\Phi(\phi)f = f \in I_u$. Thus $\ker \rho = \operatorname{Ex}(O) \subset I_u$ and we are done. □

The reason that this is a useful result is that we know a lot about the fibres of $A \rtimes G$ when $G^{(0)}/G$ is Hausdorff.

Corollary 6.13. *Suppose (A, G, α) is a separable dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. Given $u \in G^{(0)}$ the fibre $A \rtimes G(G \cdot u)$ is Morita equivalent to $A(u) \rtimes S_u$.*

Proof. Since $G^{(0)}/G$ is Hausdorff $A \rtimes G$ is a $C_0(G^{(0)}/G)$-algebra with fibres

$$A \rtimes G(G \cdot u) \cong A(G \cdot u) \rtimes G|_{G \cdot u}.$$

However, $G|_{G \cdot u}$ is a *transitive* groupoid so the result follows from Theorem 5.7. □

Thus, in the case where $G^{(0)}/G$ is Hausdorff, every irreducible representation is lifted from a fibre $A \rtimes G(G \cdot u)$, and every irreducible representation of $A \rtimes G(G \cdot u)$ comes from an irreducible representation of $A(u) \rtimes S_u$. We will show that this two stage description is nothing more than the usual induction process.

Proposition 6.14. *Suppose (A, G, α) is a separable dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. Then every irreducible representation of $A \rtimes_\alpha G$ is equivalent to one of the form $\operatorname{Ind}_{S_u}^G R$ where $u \in G^{(0)}$ and R is an irreducible representation of $A(u) \rtimes_\alpha S_u$.*

We start with a remark and a useful lemma.

Remark 6.15. Suppose we have an $A-B$-imprimitivity bimodule $\mathcal{X}$ and a representation π of B. The Rieffel induction process [RW98, Proposition 2.66] yields an induced representation $\mathcal{X}-\operatorname{Ind}\pi$ of A. It is assumed that the reader is familiar with this process. If not they may wish to use [RW98, Section 2.4] as a reference. Furthermore, the Rieffel correspondence provides a very strong link between the ideal structure and representation theory of A and the ideal structure and representation theory of B. This material can be found in [RW98, Section 3.3].

Lemma 6.16. *Suppose (A, G, α) is a separable dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. Given $u \in G^{(0)}$ let $\rho : A \rtimes G \to A(G \cdot u) \rtimes G|_{G\cdot u}$ be the extension of the restriction map on $\Gamma_c(G, r^*\mathcal{A})$. Furthermore, let $\mathcal{X}$ be the $A(G\cdot u) \rtimes G|_{G\cdot u} - A(u) \rtimes S_u$ imprimitivity bimodule from Theorem 5.7. If R is a representation of $A(u) \rtimes S_u$ then $\operatorname{Ind}_{S_u}^G R = \mathcal{X} - \operatorname{Ind}(R) \circ \rho$.*

Proof. First let us establish some notation. Let β be Haar measure on S_u. Recall that $\mathcal{X}$ is the completion of the pre-$A(G\cdot u) \rtimes G|_{G\cdot u} - A(u) \rtimes S_u$-imprimitivity bimodule $\mathcal{X}_0 = C_c(G_u, A(u))$ and that the left hand operations on $\mathcal{X}_0$ are given by

$$\begin{aligned} z \cdot g(\gamma) &= \int_{S_u} \alpha_s(z(\gamma s) g(s^{-1})) d\beta(s), \\ \langle\langle z, w\rangle\rangle_{A(u)\rtimes S_u}(s) &= \int_G z(\eta^{-1})^* \alpha_s(w(\eta^{-1}s)) d\lambda^u(\eta). \end{aligned}$$

Next let $X = s^{-1}((S_u)^{(0)}) = G_u$ and recall that the imprimitivity bimodule $\mathcal{Z}_{S_u}^G$ is the completion of $\mathcal{Z}_0 = \Gamma_c(X, s^*\mathcal{A}) = C_c(G_u, A(u))$ and carries the left hand actions

$$\begin{aligned} z \cdot g(\gamma) &= \int_{S_u} \alpha_s(z(\gamma s) g(s^{-1})) d\beta(s) \\ \langle\langle z, w\rangle\rangle_{A(u)\rtimes S_u}(s) &= \int_G z(\eta s^{-1})^* \alpha_s(w(\eta)) d\lambda_u(\eta), \\ &= \int_G z(\eta^{-1})^* \alpha_s(w(\eta^{-1}s)) d\lambda^u(\eta). \end{aligned}$$

It is a happy fact that $\mathcal{Z}_{S_u}^G$ and $\mathcal{X}$ are obviously equal as right Hilbert $A(u) \rtimes S_u$-modules.

Suppose R is a representation of $A(u) \rtimes S_u$ on $\mathcal{H}$. Recall from Theorem 6.7 that $\operatorname{Ind}_{S_u}^G R$ acts on $\mathcal{K} = \mathcal{Z}_{S_u}^G \otimes_{A(u)\rtimes S_u} \mathcal{H}$ via $\operatorname{Ind} R(f)(z \otimes h) = f \cdot z \otimes h$ where, given $f \in \Gamma_c(G, r^*\mathcal{A})$ and $z \in C_c(G_u, A(u))$,

$$f \cdot z(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta)) z(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta). \tag{6.23}$$

However, since $\mathcal{Z}_{S_u}^G = \mathcal{X}$ as right Hilbert $A(u) \rtimes S_u$-modules, the representation $\mathcal{X} - \operatorname{Ind}(R)$ also acts on $\mathcal{K}$. Furthermore the action is given by $\mathcal{X} - \operatorname{Ind}(S) f(z \otimes h) = f \odot z \otimes h$ where $f \odot z$ is the left module action of $\mathcal{X}$ and is given for $f \in \Gamma_c(G|_{G \cdot u}, r^*\mathcal{A})$ and $z \in C_c(G_u, A(u))$ by

$$f \odot z(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta)) z(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta) \tag{6.24}$$

However (6.23) and (6.24) are basically the same action and since ρ is an extension of the restriction map it is clear that for $f \in \Gamma_c(G, r^*\mathcal{A})$ we have $\operatorname{Ind}_{S_u}^G R(f) = \mathcal{X} - \operatorname{Ind}(R)(\rho(f))$ and this extends to the entire crossed product by continuity. $\square$

Actually, with this result at our disposal we are mostly done.

Proof of Proposition 6.14. Suppose (A, G, α) is a separable dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. By Proposition 6.12 $A \rtimes G$ is a $C_0(G^{(0)}/G)$-algebra. It then follows from Proposition 3.22 that any irreducible representation T is of the form $T = L \circ \rho$ where $u \in G^{(0)}$, L is an irreducible representation of $A(G \cdot u) \rtimes G|_{G \cdot u}$ and ρ is the canonical extension of the restriction map on $\Gamma_c(G, r^*\mathcal{A})$. However $A(G \cdot u) \rtimes G|_{G \cdot u}$ is Morita equivalent to $A(u) \rtimes S_u$ by Corollary 6.13. Let $\mathcal{X}$ be the bimodule implementing the equivalence and let $\widetilde{\mathcal{X}}$ be its “inverse” bimodule. Set $R = \widetilde{\mathcal{X}} - \operatorname{Ind} L$. It follows from [RW98, Theorem 3.29] that $\mathcal{X} - \operatorname{Ind} R$ is naturally equivalent to L and from [RW98, Corollary 3.32] that R is an irreducible representation. However, it follows that the representations $T = L \circ \rho$ and $\operatorname{Ind}_{S_u}^G R = \mathcal{X} - \operatorname{Ind}(R) \circ \rho$ are also equivalent and we are done. $\square$

The reason we separated out Lemma 6.16 is it allows us to easily prove the following

Proposition 6.17. *Suppose (A, G, α) is a separable dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. If R is an irreducible representation of $A(u) \rtimes S_u$ then $\operatorname{Ind}_{S_u}^G R$ is irreducible. Furthermore if L and R are both irreducible representations of $A(u) \rtimes S_u$ and $\operatorname{Ind}_{S_u}^G L$ is equivalent to $\operatorname{Ind}_{S_u}^G R$ then L is equivalent to R.*

Proof. Given (A, G, α) and R as above let $\rho : A \rtimes G \to A(G \cdot u) \rtimes G|_{G \cdot u}$ be the extension of the restriction map. Furthermore let $\mathcal{X}$ be the $A(G \cdot u) \rtimes G|_{G \cdot u} - A(u) \rtimes S_u$-imprimitivity bimodule from Theorem 5.7. Since the Rieffel correspondence preserves irreducibility $\mathcal{X} - \operatorname{Ind}(R)$ is an irreducible representation of $A(G \cdot u) \rtimes G|_{G \cdot u}$. Hence $\operatorname{Ind}_{S_u}^G R = \mathcal{X} - \operatorname{Ind}(R) \circ \rho$ must be irreducible.

Next suppose we are given two irreducible representations L and R of $A(u) \rtimes S_u$ and suppose $\operatorname{Ind}_{S_u}^G L$ and $\operatorname{Ind}_{S_u}^G R$ are equivalent. Since ρ is surjective it follows from Lemma 6.16 that $\mathcal{X} - \operatorname{Ind}(L)$ and $\mathcal{X} - \operatorname{Ind}(R)$ are equivalent. Since $\mathcal{X}$ is an imprimitivity bimodule this implies L is equivalent to R. $\square$

We are going to extend Proposition 6.14 to groupoids which satisfy the Mackey-Glimm dichotomy. For our purposes the most useful condition of Theorem 5.4 will be the fact that the orbit space is almost Hausdorff.

Definition 6.18. A, not necessarily Hausdorff, locally compact space X is said to be *almost Hausdorff* if each locally compact subspace V contains a relatively open nonempty Hausdorff subset.

The key fact we will use about almost Hausdorff spaces is the following proposition, which we cite without proof. Those readers unfamiliar with ordinals are referenced to [HJ99, Chapter 6].

Proposition 6.19 ([Wil07, Lemma 6.3])**.** *Suppose X is a, not necessarily Hausdorff, locally compact space. Then the following are equivalent.*

(a) *X is almost Hausdorff*

(b) *Every nonempty closed subspace of X has a relatively open nonempty Hausdorff subspace.*

(c) *Every closed subspace of X has a dense relatively open Hausdorff subspace.*

(d) *There is an ordinal γ and open sets $\{U_\alpha : \alpha \leq \gamma\}$ such that*

- **(i)** *$\alpha < \beta \leq \gamma$ implies that $U_\alpha \subsetneq U_\beta$,*
- **(ii)** *$\alpha < \gamma$ implies that $U_{\alpha+1} \setminus U_\alpha$ is a dense Hausdorff subspace of $X \setminus U_\alpha$,*
- **(iii)** *if $\delta \leq \gamma$ is a limit ordinal then*

$$U_\delta = \bigcup_{\alpha<\delta} U_\alpha,$$

- **(iv)** *$U_0 = \emptyset$ and $U_\gamma = X$.*

(e) *Every subspace of X has a relatively open dense Hausdorff subspace.*

The main reason we care about Proposition 6.19 is that condition (d) will allow us to build the following object.

Definition 6.20. A *composition series* in a C^*-algebra A is a family $\{I_\alpha\}_{\alpha\in\Lambda}$ of ideals I_α indexed by a segment Λ of ordinals $0 \leq \alpha \leq \gamma$ such that

(a) $I_0 = \{0\}$ and $I_\gamma = A$,

(b) $\alpha < \beta \leq \gamma$ implies $I_\alpha \subsetneq I_\beta$ and

(c) if $\delta \leq \gamma$ is a limit ordinal then

$$I_\delta = \overline{\bigcup_{\alpha<\delta} I_\alpha}.$$

Continuing the chain, the main reason we care about Definition 6.20 is the following

Lemma 6.21 ([Wil07, Lemma 8.13])**.** *Suppose $\{I_\alpha\}_{\alpha\in\Lambda}$ is a composition series for a C^*-algebra A. Then every irreducible representation π of A lives on a subquotient $I_{\alpha+1}/I_\alpha$ for some α. In other words there is an irreducible representation ρ of $I_{\alpha+1}/I_\alpha$ such that π is equal to the canonical extension of the lift of ρ to $I_{\alpha+1}$.*

Proof. Let π be an irreducible representation of A. Let $S = \{\alpha \leq \gamma : I_\alpha \not\subset \ker\pi\}$. If $\beta = \min S$ is a limit ordinal, then it follows from part (c) of Definition 6.20 that $I_\beta \subset \ker\pi$. However, this contradicts the fact that $\beta \in S$. Thus β has an immediate predecessor α. Let ρ be the factorization of $\pi|_{I_\beta}$ to I_β/I_α. Then clearly the lift of ρ to I_β is $\pi|_{I_\beta}$ and since $I_\beta \not\subset \ker\pi$ the extension of $\pi|_{I_\beta}$ to A is π. □

At this point it may be clear where we are going. If $G^{(0)}/G$ is almost Hausdorff then we will build a composition series of crossed products where the orbit space associated to the subquotients is Hausdorff. This will allow us to use Proposition 6.14 to prove the following theorem.

Theorem 6.22. *Suppose (A, G, α) is a separable groupoid dynamical system and that $G/G^{(0)}$ is a T_0 space. Then every irreducible representation of $A \rtimes_\alpha G$ is equivalent to one of the form $\operatorname{Ind}_{S_u}^G R$ where $u \in G^{(0)}$ and R is an irreducible representation of $A(u) \rtimes_\alpha S_u$.*

As before we start with a utility lemma.

Lemma 6.23. *Suppose (A, G, α) is a separable dynamical system and that $U \subset V \subset G^{(0)}$ are open G-invariant sets. Then we may identify $A(V \setminus U) \rtimes G|_{V\setminus U}$ with the subquotient $\operatorname{Ex}(V)/\operatorname{Ex}(U)$. Furthermore if $u \in U$ and R is a representation of $A(u) \rtimes S_u$ then the canonical extension of $\operatorname{Ind}_{S_u}^{G|_{V\setminus U}} R$ to $A \rtimes G$ is equal to $\operatorname{Ind}_{S_u}^G R$.*

Proof. First recall that we equip $G|_V$ with the restriction of the Haar system from G. Furthermore we equip $(G|_V)|_{V\setminus U} = G|_{V\setminus U}$ with the restriction of the Haar system coming from $G|_V$, and therefore from G. Since V is an open G-invariant set we use Proposition 5.20 to identify $A(V) \rtimes G|_V$ with the ideal $\operatorname{Ex}(V)$ via the inclusion map ι. Furthermore we also identify $A(U) \rtimes G|_U$ with $\operatorname{Ex}(U)$. Since $U \subset V$ any function which is supported in $G|_U$ must be supported in $G|_V$ as well and therefore $\operatorname{Ex}(U) \subset \operatorname{Ex}(V)$. Now U is also an open $G|_V$ invariant subset of V so that we can

also identify $A(U) \rtimes G|_U$ with its image $\operatorname{Ex}'(U)$ in $A(V) \rtimes G|_V$. We would like to see that $\iota(\operatorname{Ex}'(U)) = \operatorname{Ex}(U)$. However, this is obvious since both $\operatorname{Ex}(U)$ and $\iota(\operatorname{Ex}'(U))$ are the completion of $\Gamma_c(G|_U, r^*\mathcal{A})$ inside $A \rtimes G$. It now follows that ι factors to an isomorphism of $A(V) \rtimes G|_V / \operatorname{Ex}'(U)$ onto the subquotient $\operatorname{Ex}(V)/\operatorname{Ex}(U)$. Furthermore Theorem 5.22 implies that the restriction map ρ factors to an isomorphism of $A(V) \rtimes G|_V / \operatorname{Ex}'(U)$ with $A(V \setminus U) \rtimes G|_{V\setminus U}$. Thus $A(V \setminus U) \rtimes G|_{V\setminus U}$ is isomorphic to the subquotient $\operatorname{Ex}(V)/\operatorname{Ex}(U)$. Therefore, given a representation T of $A(V \setminus U) \rtimes G|_{V\setminus U}$ we can take its lift $T \circ \rho$ to $A(V) \rtimes G|_V$ and then extend $T \circ \rho \circ \iota^{-1}$ from $\operatorname{Ex}(V)$ to a representation of $A \rtimes G$. Of course when we are working with elements of $\Gamma_c(G|_V, r^*\mathcal{A})$ the ρ and ι^{-1} maps basically disappear so that we will usually not be this precise about viewing $A(V \setminus U) \rtimes G|_{V\setminus U}$ as a subquotient.

So suppose R is a representation of $A(u) \rtimes S_u$ on $\mathcal{H}$ for $u \in V \setminus U$. Recall that, as in the proof of Lemma 6.16, $\mathcal{Z}_{S_u}^G$ is the completion of $C_c(G_u, A(u))$ with respect to the following left operations

$$z \cdot g(\gamma) = \int_G \alpha(z(\gamma s)g(s^{-1}))d\beta(s), \tag{6.25}$$

$$\langle\langle z, w\rangle\rangle_{A(u)\rtimes S_u}(s) = \int_G z(\eta^{-1})^* \alpha_s(w(\eta^{-1}s))d\lambda^u(\eta). \tag{6.26}$$

Furthermore $\operatorname{Ind}_{S_u}^G R$ acts on $\mathcal{K} = Z_{S_u}^G \otimes_{A(u)\rtimes S_u} \mathcal{H}$ via $\operatorname{Ind}_{S_u}^G R(f)(z \otimes h) = f \cdot z \otimes h$ where, given $f \in \Gamma_c(G, r^*\mathcal{A})$, we define

$$f \cdot z(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta))g(\eta^{-1}\gamma)d\lambda^{r(\gamma)}(\eta). \tag{6.27}$$

Since the Haar system on $G|_{V\setminus U}$ is just the restriction of the Haar system of G, $\mathcal{Z}_{S_u}^{G|_{V\setminus U}}$ is also the completion of $C_c(G_u, A(u))$ with respect to the operations (6.25) and (6.26). Hence $\operatorname{Ind}_{S_u}^{G|_{V\setminus U}} R$ also acts on $\mathcal{K}$ and the action is given by $\operatorname{Ind}_{S_u}^{G|_{V\setminus U}} R(f)(z \otimes h) = f \cdot z \otimes h$ where $f \cdot z$ is defined via (6.27) on $\Gamma_c(G|_{V\setminus U}, r^*\mathcal{A})$. At this point it is clear that for $f \in \Gamma_c(G|_V, r^*\mathcal{A})$ we have

$$\operatorname{Ind}_{S_u}^G R(f) = \operatorname{Ind}_{S_u}^{G|_{V\setminus U}} R(\rho(\iota^{-1}(f))).$$

It follows that $\operatorname{Ind}_{S_u}^G R$ agrees with $\operatorname{Ind}_{S_u}^{G|_{V\setminus U}} R \circ \rho \circ \iota^{-1}$ on $\operatorname{Ex}(V)$. Hence $\operatorname{Ind}_{S_u}^G R$ is equal to the unique extension of $\operatorname{Ind}_{S_u}^{G|_{V\setminus U}} R \circ \rho \circ \iota^{-1}$ to $A \rtimes G$ and we are done. $\square$

This get us most of the way there since it shows that the process of lifting representations from a subquotient and induction are compatible. We can now prove the main result of this section.

Proof of Theorem 6.22. Since G is a second countable, locally compact Hausdorff groupoid, the fact that $G^{(0)}/G$ is T_0 implies, by Theorem 5.4, that $G^{(0)}/G$ is almost Hausdorff. Therefore there are open sets $\{V_\beta\}_{0\leq\beta\leq\gamma}$ in $G^{(0)}/G$ satisfying properties (i)-(iv) of Proposition 6.19. Let $q : G^{(0)} \to G^{(0)}/G$ be the quotient map and $U_\beta = q^{-1}(V_\beta)$ for all $0 \leq \beta \leq \gamma$. Then each U_β is an open G-invariant subset and we define $I_\beta = \mathrm{Ex}(U_\beta)$. Since $U_0 = \emptyset$ and $U_\gamma = G^{(0)}$ we must have $I_0 = \{0\}$ and $I_\gamma = A \rtimes G$. Furthermore if $\delta < \beta \leq \gamma$ then $U_\delta \subsetneq U_\beta$. Thus any function supported in $G|_{U_\delta}$ must be supported in $G|_{U_\beta}$ as well so we must have $I_\delta \subset I_\beta$. Since $U_\delta \neq U_\beta$ it is easy to build a function supported on U_β and not U_δ. Thus $I_\delta \neq I_\beta$. Finally suppose $\delta \leq \gamma$ is a limit ordinal and $f \in \Gamma_c(G|_{U_\delta}, r^*\mathcal{A})$. Because $r(\operatorname{supp} f) \subset U_\delta = \bigcup_{\beta<\delta} U_\beta$ the collection $\{U_\beta\}_{\beta<\delta}$ is an open cover of $r(\operatorname{supp} f)$. Since $r(\operatorname{supp} f)$ is compact there must be a finite subcover and since the U_β are nested this implies that there exists $\beta' < \delta$ such that $r(\operatorname{supp} f) \subset U_{\beta'}$. Hence $f \in \Gamma_c(G|_{U_{\beta'}}, r^*\mathcal{A}) \subset I_{\beta'}$. It follows that $I_\delta \subset \overline{\bigcup_{\beta<\delta} I_\beta}$. The other inclusion is trivial so that we have

$$I_\delta = \overline{\bigcup_{\beta<\delta} I_\beta}.$$

Thus $\{I_\beta\}$ is a composition series for $A \rtimes G$.

Now suppose L is a irreducible representation of $A \rtimes G$. Lemma 6.21 implies that there exists β such that L lives on $I_{\beta+1}/I_\beta$. In other words, there is an irreducible representation T of $I_{\beta+1}/I_\beta$ such that L is the unique extension of the lift of T. Next, Lemma 6.23 implies that we can identify $I_{\beta+1}/I_\beta$ with $A(U_{\beta+1} \setminus U_\beta) \rtimes G|_{U_{\beta+1}\setminus U_\beta}$. Furthermore $(U_{\beta+1} \setminus U_\beta)/G = V_{\beta+1} \setminus V_\beta$ is Hausdorff so that by Proposition 6.14 there exists $u \in U_{\beta+1} \setminus U_\beta$ and an irreducible representation R of $A(u) \rtimes S_u$ such that T is equivalent to $R' = \mathrm{Ind}_{S_u}^{G|_{U_{\beta+1}\setminus U_\beta}} R$. Hence the extension of T to $A \rtimes G$, which is L, is equivalent to the extension of R' to $A \rtimes G$, which is $\mathrm{Ind}_{S_u}^G R$ by Lemma 6.23. $\square$

As before, we separated out Lemma 6.23 so that we could prove the following

Proposition 6.24. *Suppose (A, G, α) is a separable dynamical system and the orbit space $G^{(0)}/G$ is T_0. If R is an irreducible representation of $A(u) \rtimes S_u$ then $\mathrm{Ind}_{S_u}^G R$ is irreducible. Furthermore, if R and L are both irreducible representations of $A(u) \rtimes S_u$ and $\mathrm{Ind}_{S_u}^G R$ is equivalent to $\mathrm{Ind}_{S_u}^G L$ then R is equivalent to L.*

Proof. Suppose R is an irreducible representation of $A(u) \rtimes S_u$. Using Theorem 5.4 $G^{(0)}/G$ must be locally Hausdorff. Let $\{V_\beta\}$ be as in Proposition 6.19. Consider $\Gamma = \{\beta \leq \gamma : G \cdot u \in V_\beta\}$. If $\delta = \min \Gamma$ is a limit ordinal then $G \cdot u \in \bigcup_{\beta<\delta} V_\beta$. However this implies $G \cdot u \in V_\beta$ for some $\beta < \delta$. This is a contradiction. It follows that δ has an immediate predecessor σ and $G \cdot u \in V_\delta \setminus V_\sigma$. Let $q : G^{(0)} \to G^{(0)}/G$ be the quotient map, $U_\delta = q^{-1}(V_\delta)$ and $U_\sigma = q^{-1}(V_\sigma)$. Then $u \in U_\delta \setminus U_\sigma$ and since

$V_\delta \setminus V_\sigma$ is Hausdorff we can use Proposition 6.17 to conclude that $R' = \operatorname{Ind}_{S_u}^{G|_{U_\delta \setminus U_\sigma}} R$ is irreducible. However it follows that the extension of R' to $A \rtimes G$ is irreducible and by Lemma 6.23 this is exactly $\operatorname{Ind}_{S_u}^{G} R$.

Now suppose R and L are both irreducible representations on $A(u) \rtimes S_u$ and that $\operatorname{Ind}_{S_u}^{G} R$ is equivalent to $\operatorname{Ind}_{S_u}^{G} L$. This implies that their factorizations to the subquotient $A(U_\delta \backslash U_\sigma) \rtimes G|_{U_\delta \backslash U_\sigma}$ are equivalent. It follows from Lemma 6.23 that these factorizations are $\operatorname{Ind}_{S_u}^{G|_{U_\delta \backslash U_\sigma}} R$ and $\operatorname{Ind}_{S_u}^{G|_{U_\delta \backslash U_\sigma}} L$ is respectively. Hence, Proposition 6.17 implies that R and L must be equivalent. □

Remark 6.25. It would be tempting, in light of Theorem 6.22, to say that every representation of $A \rtimes G$ is induced from a stabilizer. Unfortunately, this notion has a conflicting definition in [Wil07, Definition 8.10]. The problem lies in the meaning of the word stabilizer. In [Wil07] the stabilizers are the stabilizer subgroups with respect to the action of G on $\operatorname{Prim} A$. In Theorem 6.22 the stabilizers are with respect to the action of G on its unit space, which may be larger. Of course, when A has Hausdorff spectrum equal to $G^{(0)}$ these two notions match up. Furthermore when A has Hausdorff spectrum it is not hard to show, using Example 4.7, that Theorem 6.22 generalizes [Wil07, Theorem 8.16]. It's also worth pointing out that this is the only way we can view the results of this section as generalizing the group case. Theorem 6.22 is trivial if we take the naive approach and treat groups as groupoids with a single unit.

Remark 6.26. Generalizing this result to groupoids which do not satisfy the Mackey-Glimm Dichotomy is difficult. For group crossed products the result is known as the Gootman-Rosenberg-Sauvageot (GRS) theorem. The method of attack was developed by Sauvageot in [Sau78, Sau79] and the complete solution was given by Gootman and Rosenberg in [GR79]. The result is also proved in [Wil07, Chapter 9]. For groupoid C^*-algebras, the corresponding result is proved in [IW08] and for general groupoid crossed products the question is still open.

6.3 Crossed Products with Abelian Isotropy

Theorem 6.22 is a nice enough result, but if we want to study the fine structure of $A \rtimes G$ we need to consider more than just individual representations. This next proposition adds a topological component to the results of the last section.

Proposition 6.27. *Suppose (A, G, α) is a separable dynamical system. Furthermore, suppose that the isotropy subgroupoid S varies continuously and that $G^{(0)}/G$ is a T_0 space. Then $\Phi : (A \rtimes S)^\wedge \to (A \rtimes G)^\wedge$ given by $\Phi(R) = \operatorname{Ind}_S^G R$ is a continuous surjection.*

Recall that $A \rtimes S$ is a $C_0(G^{(0)})$-algebra and that restriction factors to an isomorphism of $A \rtimes S(u)$ with $A(u) \rtimes S_u$. The main difficulty is to show that induction respects this fibering.

Lemma 6.28. *Suppose (A, G, α) is a separable dynamical system and that the isotropy subgroupoid S varies continuously. Given $u \in G^{(0)}$ and a representation R of $A(u) \rtimes S_u$ let $\rho : A \rtimes S \to A(u) \rtimes S_u$ be given on $\Gamma_c(S, p^*\mathcal{A})$ by restriction. Then $\operatorname{Ind}_{S_u}^G R$ is equivalent to $\operatorname{Ind}_S^G(R \circ \rho)$.*

Proof. First, since S is continuously varying, it is a closed subgroupoid of G with its own Haar system, which we call β. Let $u \in G^{(0)}$ and R be a representation of $A(u) \rtimes S_u$ on $\mathcal{H}$. Consider the right Hilbert $A(u) \rtimes S_u$-module $\mathcal{Z}_{S_u}^G$ associated to $\operatorname{Ind}_{S_u}^G R$. It follows from Proposition 6.3 that $\mathcal{Z}_{S_u}^G$ is the completion of $C_c(G_u, A(u))$ with respect to the right actions

$$z \cdot g(\gamma) = \int_{S_u} \alpha_s(z(\gamma s)g(s^{-1}))d\beta^u(s)$$
$$\langle\langle z, w\rangle\rangle_{A(u)\rtimes S_u}(s) = \int_G z(\eta s^{-1})^* \alpha_s(w(\eta))d\lambda_u(\eta).$$

Furthermore, $\operatorname{Ind}_{S_u}^G R$ acts on $\mathcal{Z}_{S_u}^G \otimes_{A(u)\rtimes S_u} \mathcal{H}$ which, as in Remark 6.6, is the completion of $C_c(G_u, A(u)) \odot \mathcal{H}$ with respect to the inner product characterized by

$$(f \otimes h, g \otimes k) = (R(\langle\langle g, f\rangle\rangle_{A(u)\rtimes S_u})h, k).$$

Now consider the right Hilbert $A \rtimes S$-module $\mathcal{Z}_S^G$ associated to $\operatorname{Ind}_S^G(R \circ \rho)$. It follows from Proposition 6.3 that $\mathcal{Z}_S^G$ is the completion of $\Gamma_c(G, s^*\mathcal{A})$ with respect to the operations

$$z \cdot g(\gamma) = \int_S \alpha_s(z(\gamma s)g(s^{-1}))d\beta^{s(\gamma)}(s)$$
$$\langle\langle z, w\rangle\rangle_{A\rtimes S}(s) = \int_G z(\eta s^{-1})^* \alpha_s(w(\eta))d\lambda_{p(s)}(\eta).$$

Furthermore $\operatorname{Ind}_S^G(R \circ \rho)$ acts on $\mathcal{Z}_S^G \otimes_{A\rtimes S} \mathcal{H}$ which is the completion of $\Gamma_c(G, s^*\mathcal{A}) \odot \mathcal{H}$ with respect to the inner product characterized by

$$(f \otimes h, g \otimes k) = (R(\rho(\langle\langle g, f\rangle\rangle_{A\rtimes S}))h, k)$$

We would like to define a unitary map $U : \mathcal{Z}_S^G \otimes \mathcal{H} \to \mathcal{Z}_{S_x}^G \otimes \mathcal{H}$. Start by letting $\pi : \Gamma_c(G, s^*\mathcal{A}) \to C_c(G_u, A(u))$ be given by restriction. Since π is clearly linear we can define $U : \Gamma_c(G, s^*\mathcal{A}) \odot \mathcal{H} \to C_c(G_u, A(u)) \odot \mathcal{H}$ on elementary tensors

by $U(f \otimes h) = \pi(f) \otimes h$. It is clear enough that given $f, g \in \Gamma_c(G, s^*\mathcal{A})$ we have $\rho(\langle\langle f, g\rangle\rangle_{A \rtimes S}) = \langle\langle \pi(f), \pi(g)\rangle\rangle_{A(u) \rtimes S_u}$ so that

$$\begin{aligned}(f \otimes h, g \otimes k) &= (R(\rho(\langle\langle g, f\rangle\rangle_{A \rtimes S}))h, k)\\ &= (R(\langle\langle \pi(g), \pi(f)\rangle\rangle_{A(u) \rtimes S_u})h, k)\\ &= (\pi(f) \otimes h, \pi(g) \otimes k)\\ &= (U(f \otimes h), U(f \otimes k)).\end{aligned}$$

Thus U is isometric on $\Gamma_c(G, s^*\mathcal{A}) \odot \mathcal{H}$ and we can extend it to an isometry on $\mathcal{Z}_S^G \otimes \mathcal{H}$. We need to show that U is surjective. Suppose $f \in C_c(G_u)$ and $a \in A(u)$. Choose $b \in A$ such that $b(u) = a$ and extend f to a function $g \in C_c(G)$. Then $g \otimes b \in \Gamma_c(G, s^*\mathcal{A})$ and given $\gamma \in G_u$ we clearly have $\pi(g \otimes b)(\gamma) = f(\gamma)a = f \otimes a(\gamma)$. Thus $\operatorname{ran} \pi$ contains all of the elementary tensors in $C_c(G_u, A(u))$ and as such is dense in the inductive limit topology. Now suppose $z_i \to z$ with respect to the inductive limit topology in $C_c(G_u, A(u))$. Let K be some compact set which eventually contains the supports of the z_i. Then

$$\begin{aligned}&\|\langle\langle z_i, z_i\rangle\rangle_{A(u) \rtimes S_u}(s) - \langle\langle z, z\rangle\rangle_{A(u) \rtimes S_u}(s)\|\\ &\qquad\le \int_G \|z_i(\eta s^{-1})^* \alpha_s(z_i(\eta)) - z(\eta s^{-1})^* \alpha_s(z(\eta))\| d\lambda_u(\eta)\\ &\qquad\le \int_G \|z_i(\eta s^{-1}) - z(\eta s^{-1})\| \|z_i(\eta)\| + \|z(\eta s^{-1})\| \|z_i(\eta) - z(\eta)\| d\lambda_u(\eta)\\ &\qquad\le \|z_i - z\|_\infty (\|z_i\|_\infty + \|z\|_\infty)\lambda^u(K)\end{aligned}$$

Since $\{\|z_i\|_\infty\}$ is bounded this shows that $\langle\langle z_i, z_i\rangle\rangle \to \langle\langle z, z\rangle\rangle$ uniformly. Furthermore $\operatorname{supp}\langle\langle z_i, z_i\rangle\rangle$ is eventually contained in $K^{-1}K$, which is compact. Thus $\langle\langle z_i, z_i\rangle\rangle \to \langle\langle z, z\rangle\rangle$ with respect to the inductive limit topology and hence

$$R(\langle\langle z_i, z_i\rangle\rangle_{A(u) \rtimes S_u}) \to R(\langle\langle z, z\rangle\rangle_{A(u) \rtimes S_u}).$$

Now suppose $z \in C_c(G_u, A(u))$ and $h \in \mathcal{H}$. Choose $z_i = \pi(w_i)$ such that $z_i \to z$ with respect to the inductive limit topology. Then, by the above, since $z_i - z \to 0$ with respect to the inductive limit topology,

$$\|U(w_i \otimes h) - z \otimes h\|^2 = \|(z_i - z) \otimes h\|^2 = (R(\langle\langle z_i - z, z_i - z\rangle\rangle_{A(u) \rtimes S_u})h, h) \to 0.$$

It follows that $\operatorname{ran} U$ is dense in $\mathcal{Z}_{S_u}^G \otimes \mathcal{H}$ and that U is a unitary.

The last step is to show that U intertwines $\operatorname{Ind}_S^G(R \circ \rho)$ and $\operatorname{Ind}_{S_u}^G R$. According

to Theorem 6.7 $\operatorname{Ind}_{S_u}^R$ acts via $\operatorname{Ind}_{S_u}^G R(f)(z \otimes h) = f \cdot z \otimes h$ where

$$f \cdot z(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta)) z(\eta^{-1}\gamma) d\lambda^{r(\gamma)}(\eta). \tag{6.28}$$

and that $\operatorname{Ind}_S^G(R \circ \rho)$ also acts via $\operatorname{Ind}_S^G(R \circ \rho)(f)(z \otimes h) = f \cdot z \otimes h$ where $f \cdot z$ is still given by (6.28). It is clear that for $z \in \Gamma_c(G, s^*\mathcal{A})$ we have $f \cdot \pi(z)(\gamma) = \pi(f \cdot z)(\gamma)$. Thus on $\Gamma_c(G, s^*\mathcal{A}) \odot \mathcal{H}$

$$\operatorname{Ind}_S^G(R \circ \rho)(f) U(z \otimes h) = f \cdot \pi(z) \otimes h = \pi(f \cdot z) \otimes h = U \operatorname{Ind}_{S_u}^G R(f)(z \otimes h).$$

This suffices to show that U intertwines $\operatorname{Ind}_S^G(R \circ \rho)$ and $\operatorname{Ind}_{S_u}^G R$. $\square$

Remark 6.29. In light of how natural the unitary intertwining $\operatorname{Ind}_{S_u}^G R$ and $\operatorname{Ind}_S^G(R \circ \rho)$ is we will often confuse the two. Furthermore, since every irreducible representation of $A \rtimes S$ is lifted from a fibre via restriction we will feel free to use the notation $\operatorname{Ind}_S^G R$ even when R is an irreducible representation of $A(u) \rtimes S_u$. Furthermore we will interpret $\operatorname{Ind}_S^G R$ as either $\operatorname{Ind}_{S_u}^G R$ or as $\operatorname{Ind}_S^G(R \circ \rho)$ as we see fit. We trust the reader will forgive the author for these abuses.

The advantage of viewing the induction as occurring on S is that induction from a fixed algebra is a continuous process.

Proof of Proposition 6.27. Since S is a continuously varying group bundle we know $A \rtimes S$ exists and that every irreducible representation is lifted from a fibre. In particular every irreducible representation is of the form $R \circ \rho$ where R is an irreducible representation of $A(u) \rtimes S_u$ for some u and ρ is the canonical extension of the restriction map. Since $G^{(0)}/G$ is T_0 we may use Proposition 6.24 to conclude that $\operatorname{Ind}_{S_u}^G R$, and hence $\operatorname{Ind}_S^G(R \circ \rho)$ is irreducible. Thus $\Phi(R) = \operatorname{Ind}_S^G R$ is a well defined map from $(A \rtimes S)^\wedge$ into $(A \rtimes G)^\wedge$. Furthermore Theorem 6.22 tells us that every irreducible representation is (equivalent to one) of the form $\operatorname{Ind}_{S_u}^G R$ so that Φ is surjective. Finally, we show that Φ is continuous. This follows from the general theory of Rieffel induction. In particular [RW98, Corollary 3.35] and the definition of Ind_S^G implies that the map

$$\ker R \mapsto \operatorname{Ind}_{S_u}^G \ker R = \ker \operatorname{Ind}_S^G R$$

is continuous. Since the topology on the spectrum of a C^*-algebra is inherited from the Jacobson topology on the space of primitive ideals it is straightforward to show that Φ must be continuous. $\square$

6.3.1 Groupoid Actions

At this point we begin to transition over to the use of abelian isotropy subgroups, an assumption that will stick with us. The reason is because this assumption will allow us to identify the equivalence classes determined by Φ. It is worth pointing out that the odd result here or there may extend to the nonabelian case.

Proposition 6.30 ([PSMW96, Lemma 4.1])**.** *Suppose G is a second countable locally compact groupoid and that the isotropy subgroupoid S is abelian and varies continuously. Then there is a continuous S-invariant homomorphism ω from G to $\mathbb{R}^+$ such that for all $f \in C_c(S)$*

$$\int_S f(s) d\beta^{r(\gamma)}(s) = \omega(\gamma) \int_S f(\gamma s \gamma^{-1}) d\beta^{s(\gamma)}(s). \tag{6.29}$$

Proof. Let β be a Haar system for S. Given $\gamma \in G$ consider the map $\phi^\gamma : S_{s(\gamma)} \to S_{r(\gamma)}$ defined by $\phi^\gamma(s) = \gamma s \gamma^{-1}$. It is clear that ϕ^γ is a group isomorphism so that we can push forward the Haar measure $\beta^{s(\gamma)}$ to a Haar measure on $S_{r(\gamma)}$ defined for $f \in C_c(S)$ by

$$\phi^\gamma_* \beta^{s(\gamma)}(f) = \int_S f(\gamma s \gamma^{-1}) d\beta^{s(\gamma)}(s).$$

However, Haar measure is unique up to a scalar multiple so there exists $\omega(\gamma) \in \mathbb{R}^+$ such that $\beta^{r(\gamma)} = \omega(\gamma)\phi^\gamma_* \beta^{s(\gamma)}$. It is clear that ω is the map we are looking for. Furthermore, it is easy to show that if γ and η are composable then $\phi^{\gamma\eta} = \phi^\gamma \circ \phi^\eta$ so that

$$\phi^{\gamma\eta}_* \beta^{s(\eta)} = \phi^\gamma_* \phi^\eta_* \beta^{s(\eta)}.$$

It follows that $\omega(\gamma\eta) = \omega(\gamma)\omega(\eta)$. Finally, if $s \in S_{s(\gamma)}$ then, since the stabilizers are abelian, we have $\phi^{\gamma s} = \phi^\gamma$ and ω is S-invariant on the left. A similar argument shows that it is invariant on the right.

Now we show that ω is continuous. This portion of the proof is taken from [PSMW96, Lemma 4.1]. Suppose to the contrary that there exists $\gamma_n \to \gamma_0$ such that $|\omega(\gamma_n) - \omega(\gamma_0)| \geq \epsilon > 0$ for all n. We can certainly choose $f \in C_c(S)$ such that $\int_S f d\beta^{r(\gamma_0)} = 1$. Thus $\int_S f d\beta^{r(\gamma_n)}$ is eventually nonzero. We claim that we may as well assume that $s(\gamma_n) \neq s(\gamma_0)$ for all $n > 0$. If not, then we can pass to a subsequence, relabel, and assume that $s(\gamma_n) = s(\gamma_0) = u$ for all n. Now suppose $\delta > 0$ and that there exists s_n such that

$$|f(\gamma_n s_n \gamma_n^{-1}) - f(\gamma_0 s_n \gamma_0^{-1})| \geq \delta \tag{6.30}$$

for all $n > 0$. For (6.30) to hold we must have either have $\gamma_n s_n \gamma_n^{-1} \in \operatorname{supp} f$ infinitely often or $\gamma_0 s_n \gamma_0 \in \operatorname{supp} f$ infinitely often. Either way we can pass to a subnet and find

s such that $s_n \to s$. However we then have

$$f(\gamma_n s_n \gamma_n^{-1}) \to f(\gamma_0 s \gamma_0^{-1})$$
$$f(\gamma_0 s_n \gamma_0^{-1}) \to f(\gamma_0 s \gamma_0^{-1})$$

which contradicts (6.30). This shows that $f \circ \phi^{\gamma_n} \to f \circ \phi^{\gamma_0}$ uniformly on $C_c(S_u)$. Next, let U be some compact neighborhood of γ_0. Then eventually $\gamma_n \in U$ and $\operatorname{supp}(f \circ \phi^{\gamma_n})$ is contained in the compact set

$$\{\gamma^{-1} s \gamma : \gamma \in U, s \in \operatorname{supp} f, s(\gamma) = p(s), r(\gamma) = u\}.$$

Thus $f \circ \phi^{\gamma_n} \to f \circ \phi^{\gamma_0}$ with respect to the inductive limit topology and

$$\phi_*^{\gamma_n} \beta^u(f) = \int_S f(\gamma_n s \gamma_n^{-1}) d\beta^u(s) \to \phi_*^{\gamma_0} \beta^u(f) = \int_S f(\gamma_0 s \gamma_0^{-1}) d\beta^u(s).$$

Therefore

$$\omega(\gamma_n)^{-1} = (\beta^{r(\gamma_n)}(f))^{-1} \phi_*^{\gamma_n} \beta^{s(\gamma_n)}(f) \to \omega(\gamma_0)^{-1} = (\beta^{r(\gamma_0)}(f))^{-1} \phi_*^{\gamma_n} \beta^{s(\gamma_0)}(f) \tag{6.31}$$

which leads to a contradiction.

This proves our claim so that we may assume $s(\gamma_n) \neq s(\gamma_0)$ for all $n > 0$. By passing to a subsequence and relabeling we can also assume that $s(\gamma_n) \neq s(\gamma_m)$ for all $n \neq m$. Then $C = p^{-1}(\{s(\gamma_n)\}_{n=0}^{\infty})$ is closed in S and we can define ι on C by $\iota(s) = n$ if and only if $p(s) = s(\gamma_n)$. Then it is straightforward to show that the function

$$F_0(s) = f(\gamma_{\iota(s)} s \gamma_{\iota(s)}^{-1})$$

is continuous and compactly supported on C. Therefore we can find an extension $F \in C_c(S)$. However, we then have

$$\phi_*^{\gamma_n} \beta^{s(\gamma_n)}(f) = \int_S F(s) d\beta^{s(\gamma_n)}(s) \to \phi_*^{\gamma_0} \beta^{s(\gamma_0)}(f) = \int_S F(s) d\beta^{s(\gamma_0)}(s)$$

and we obtain a contradiction just as in (6.31). □

We can now perform a construction which is, in many ways, interesting in its own right, even though we will only make use of it indirectly.

Proposition 6.31. *Suppose (A, G, α) is a separable dynamical system and that the isotropy subgroupoid S is abelian and varies continuously. Then there is an action of G on $A \rtimes_\alpha S$ defined by the collection $\{\delta_\gamma\}_{\gamma \in G}$ where, for $f \in C_c(S_{s(\gamma)}, A(s(\gamma)))$,*

$$\delta_\gamma(f)(s) = \omega(\gamma)^{-1} \alpha_\gamma(f(\gamma^{-1} s \gamma)).$$

Proof. First, recall that $A \rtimes S$ is a $C_0(G^{(0)})$-algebra so that it makes sense to define a groupoid action. Furthermore, as usual we will use the restriction map to identify the fibres with $A(u) \rtimes S_u$. It is easy to see that δ_γ maps $C_c(S_{s(\gamma)}, A(s(\gamma)))$ into $C_c(S_{r(\gamma)}, A(r(\gamma)))$ and that δ_γ is continuous with respect to the inductive limit topology (on both algebras). We will show it is a $*$-homomorphism. Given $f, g \in C_c(S_{s(\gamma)}, A(s(\gamma)))$ we have

$$\begin{aligned}
\delta_\gamma(f * g)(s) &= \omega(\gamma)^{-1}\alpha_\gamma(f * g(\gamma^{-1}s\gamma)) \\
&= \int_S \omega(\gamma)^{-1}\alpha_\gamma(f(t)\alpha_t(g(t^{-1}\gamma^{-1}s\gamma)))d\beta^{s(\gamma)}(\eta) \\
&= \int_S \omega(\gamma)^{-2}\alpha_\gamma(f(\gamma^{-1}t\gamma)\alpha_{\gamma^{-1}t\gamma}(g(\gamma^{-1}t^{-1}s\gamma)))d\beta^{r(\gamma)}(t) \\
&= \int_S \delta_\gamma(f)(t)\alpha_t(\delta_\gamma(g)(t^{-1}s))d\beta^{r(\gamma)}(t) \\
&= \delta_\gamma(f) * \delta_\gamma(g)(s),
\end{aligned}$$

as well as

$$\begin{aligned}
\delta_\gamma(f^*)(s) &= \omega(\gamma)^{-1}\alpha_\gamma(f^*(\gamma^{-1}s\gamma)) \\
&= \omega(\gamma)^{-1}\alpha_\gamma(\alpha_{\gamma^{-1}s\gamma}(f(\gamma^{-1}s^{-1}\gamma)^*)) \\
&= \alpha_s(\omega(\gamma)^{-1}\alpha_\gamma(f(\gamma^{-1}s^{-1}\gamma))^*) \\
&= \alpha_s(\delta_\gamma(f)(s^{-1})^*) = \delta_\gamma(f)^*(s).
\end{aligned}$$

Since δ_γ is a $*$-homomorphism which is continuous with respect to the inductive limit topology Corollary 3.134 shows that it is bounded and extends to $A(u) \rtimes S_u$. Next we observe that $\delta_u = \mathrm{id}$ and that for composable γ and η

$$\begin{aligned}
\delta_\gamma(\delta_\eta(f))(s) &= \omega(\gamma)^{-1}\omega(\eta)^{-1}\alpha_\gamma(\alpha_\eta(f(\eta^{-1}\gamma^{-1}s\gamma\eta))) \\
&= \omega(\gamma\eta)^{-1}\alpha_{\gamma\eta}(f((\gamma\eta)^{-1}s(\gamma\eta))) \\
&= \delta_{\gamma\eta}(f)(s).
\end{aligned}$$

This not only shows that δ_γ is an isomorphism with inverse $\delta_{\gamma^{-1}}$, it also shows that δ preserves the groupoid operations. All we need to do now is show that the action is continuous.

Let $\mathcal{E}$ be the upper-semicontinuous bundle associated to $A \rtimes S$. Suppose $\gamma_n \to \gamma_0$ and that $a_n \to a_0$ in $\mathcal{E}$ such that $s(\gamma_n) = p(a_n) = u_n$ for all $n \geq 0$. Fix $\epsilon > 0$ and let $v_n = r(\gamma_n)$ for all $n \geq 0$. First, choose $b \in A \rtimes S$ such that $b(u_0) = a_0$. Next, using the fact that $\Gamma_c(S, p^*\mathcal{A})$ is dense in $A \rtimes S$, we can choose $f \in \Gamma_c(S, p^*\mathcal{A})$ such that

$\|f - b\| < \epsilon/2$. In particular

$$\|f(u) - b(u)\| < \epsilon/2$$

for all $u \in U$ where we recall that $f(u)$ denotes the restriction of f to S_u. We make the following claim.

Claim. If $f \in \Gamma_c(S, p^*\mathcal{A})$ and $\gamma_n \to \gamma_0$ as above then $\delta_{\gamma_n}(f(u_n)) \to \delta_{\gamma_0}(f(u_0))$ in $\mathcal{E}$.

Proof of Claim. As in the proof that ω is continuous, we first suppose that $v_n = v_0$ infinitely often. Then we can pass to a subsequence, relabel, and assume $v_n = v_0$ for all n. Suppose that we can pass to a subsequence such that for each $n > 0$ there exists $s_n \in S_{v_0}$ such that

$$\|\delta_{\gamma_n}(f(u_n))(s_n) - \delta_{\gamma_0}(f(u_0))(s_n)\| \geq \epsilon > 0 \tag{6.32}$$

for all $n > 0$. If this is to hold then we either must have $\gamma_n^{-1} s_n \gamma_n \in \operatorname{supp} f$ infinitely often or $\gamma_0^{-1} s_n \gamma_0 \in \operatorname{supp} f$ infinitely often. In either case we may pass to a subsequence, multiply by the appropriate groupoid elements, and find s_0 such that $s_n \to s_0$. However we then have

$$\begin{aligned} &f(\gamma_n s_n \gamma_n^{-1}) \to f(\gamma_0 s_0 \gamma_0^{-1}), \quad \text{and} \\ &f(\gamma_0 s_n \gamma_0^{-1}) \to f(\gamma_0 s_0 \gamma_0^{-1}). \end{aligned}$$

Since α and ω are continuous, it follows that $\delta_{\gamma_n}(f(u_n))(s_n)$ and $\delta_{\gamma_0}(f(u_0))(s_n)$ both converge to $\delta_{\gamma_0}(f(u_0))(s_0)$. This contradicts (6.32). As a result $\delta_{\gamma_n}(f(u_n))$ must converge to $\delta_{\gamma_0}(f(u_0))$ uniformly. Let U be a compact neighborhood of γ_0. Eventually $\gamma_n \in U$ and therefore eventually $\operatorname{supp} \delta_{\gamma_n}(f(u_n))$ is contained in the compact set

$$\{\gamma^{-1} s \gamma : \gamma \in U, s \in \operatorname{supp} f, s(\gamma) = p(s), r(\gamma) = v_0\}.$$

Thus $\delta_{\gamma_n}(f(u_n)) \to \delta_{\gamma_0}(f(u_0))$ with respect to the inductive limit topology and thus in $A(v_0) \rtimes S_{v_0} \subset \mathcal{E}$.

Next, suppose that we may remove an initial segment and assume that $v_n \neq v_0$ for all $n > 0$. Furthermore, we may pass to a subsequence, relabel, and assume that $v_n \neq v_m$ for all $n \neq m$. Then $C = p^{-1}(\{v_n\}_{n=0}^{\infty})$ is closed in S and we can define ι on C by $\iota(s) = n$ if and only if $p(s) = v_n$. We would like to show that

$$F_0(s) = \delta_{\gamma_{\iota(s)}}(f)(s) = \omega(\gamma_{\iota(s)})^{-1} \alpha_{\gamma_{\iota(s)}}(f(\gamma_{\iota(s)}^{-1} s \gamma_{\iota(s)}))$$

defines a compactly supported continuous function on C. Suppose $s_i \to s$. If $\iota(s_i)$ is eventually constant then the convergence of $F_0(s_i) \to F_0(s)$ is easy. However, it is also easy if $\iota(s_i) \to \infty$ because in this case we just use the fact that $\gamma_i \to \gamma_0$ and

that all the different components are continuous. Furthermore it is straightforward to check that F_0 has support contained in the compact set

$$\{\eta s \eta^{-1} : \eta \in \{\gamma_n\}_{n=0}^{\infty}, s \in \operatorname{supp} f, s(\eta) = p(s)\}.$$

Now, $K = \{v_n\}_{n=0}^{\infty}$ is an S-invariant closed subset of $G^{(0)}$ so that we may use Proposition 5.24 to conclude that $A \rtimes S(K)$ is isomorphic to $A(K) \rtimes C$. Thus, since the restriction map is a surjective homomorphism of $A \rtimes S$ onto $A \rtimes S(K)$ by Proposition 5.14, and since $F_0 \in \Gamma_c(C, p^*\mathcal{A})$, there must be some $F \in A \rtimes S$ such that $F(v_n) = F_0(v_n)$ for all $n \geq 0$. In particular, since F is a continuous section of $\mathcal{E}$, we have $F_0(v_n) \to F_0(v_0)$. However we clearly constructed F_0 so that $F_0(v_n) = \delta_{\gamma_n}(f(u_n))$ for all $n \geq 0$. This proves our claim. □

Thus $\delta_{\gamma_n}(f(u_n)) \to \delta_{\gamma_0}(f(u_0))$. Furthermore we have

$$\|\delta_{\gamma_0}(f(u_0)) - \delta_{\gamma_0}(a_0)\| = \|f(u_0) - b(u_0)\| < \epsilon/2 < \epsilon.$$

by construction. Since both $a_n \to a_0$ and $b(u_n) \to a_0$ it follows that $\|a_n - b(u_n)\| \to 0$ so that eventually

$$\|\delta_{\gamma_n}(f(u_n)) - \delta_{\gamma_n}(a_n)\| \leq \|f(u_n) - b(u_n)\| + \|b(u_n) - a_n\| < \epsilon.$$

It now follows from the last part of Proposition 3.2 that $\delta_{\gamma_n}(a_n) \to \delta_{\gamma_0}(a_0)$. Hence the action is continuous and we are done. □

We get the following immediate and important corollary from Proposition 4.39. Looking ahead, this corollary lays the foundation for our identification of the equivalence classes determined by Φ.

Corollary 6.32. *Suppose (A, G, α) is a separable dynamical system and that the stabilizer subgroupoid S is abelian and continuously varying. Then the action δ induces an action of G on $(A \rtimes S)^\wedge$ such that $\gamma \cdot R = R \circ \delta_\gamma^{-1}$ for all $\gamma \in G$ and $R \in (A(s(\gamma)) \rtimes S_{s(\gamma)})^\wedge$.*

Of course, we would like to find a covariant decomposition for the above action.

Proposition 6.33. *Suppose (A, G, α) is a separable dynamical system and that the isotropy subgroupoid S is abelian and continuously varying. If $R = \pi \rtimes U$ is a representation of $A(u) \rtimes S_u$ then $\gamma \cdot R = \rho \rtimes V$ where*

$$\rho(a) = \pi(\alpha_\gamma^{-1}(a)), \quad \textit{and} \quad V_s = U_{\gamma^{-1} s \gamma}. \tag{6.33}$$

Proof. Suppose we are given (A, G, α) and R as above with $\gamma \in G$ such that $s(\gamma) = u$. Let $v = r(\gamma)$. Since $A(v) \rtimes S_v$ is a group crossed product we have a lot of technology

at our disposal. In particular we know that there must be a covariant representation (ρ, V) such that $\gamma \cdot R = \rho \rtimes V$. It follows from [Wil07, Proposition 2.34] that $\rho = \overline{R} \circ \iota_{A(v)}$ and $V = \overline{R} \circ \iota_{S_v}$ where $\iota_{A(v)}$ and ι_{S_v} are the canonical maps given by

$$\iota_{A(v)}(a)f(s) = af(s), \qquad\qquad \iota_{S_v}(s)f(t) = \alpha_s(f(s^{-1}t)).$$

We compute that

$$\begin{aligned}
V_s(\gamma \cdot R)(f)h &= \gamma \cdot R(\iota_{S_v}(s)f)h = R(\delta_\gamma^{-1}(\iota_{S_v}(s)f))h \\
&= \int_S \pi(\delta_\gamma^{-1}(\iota_{S_v}(s)f)(t))U_t h d\beta^u(t) \\
&= \int_S \omega(\gamma)\pi(\alpha_\gamma^{-1}(\iota_{S_v}(s)f(\gamma t\gamma^{-1})))U_t h d\beta^u(t) \\
&= \int_S \omega(\gamma)\pi(\alpha_{\gamma^{-1}s}(f(s^{-1}\gamma t\gamma^{-1})))U_t h d\beta^u(t) \\
&= \int_S \omega(\gamma)U_{\gamma^{-1}s\gamma}\pi(\alpha_\gamma^{-1}(f(\gamma(\gamma^{-1}s^{-1}\gamma)t\gamma^{-1})))U_{\gamma^{-1}s^{-1}\gamma t} h d\beta^u(t).
\end{aligned}$$

Using the fact that β^u is left invariant we obtain

$$\begin{aligned}
V_s(\gamma \cdot R)(f)h &= U_{\gamma^{-1}s\gamma}\int_S \omega(\gamma)\pi(\alpha_\gamma^{-1}(f(\gamma t\gamma^{-1})))U_t h d\beta^u(t) \\
&= U_{\gamma^{-1}s\gamma}\int_S \pi(\delta_\gamma^{-1}(f)(t))U_t h d\beta^u(t) \\
&= U_{\gamma^{-1}s\gamma}R(\delta_\gamma^{-1}(f))h = U_{\gamma^{-1}s\gamma}(\gamma \cdot R)(f)h.
\end{aligned}$$

Since $\gamma \cdot R$ is nondegenerate this shows that $V_s = U_{\gamma^{-1}s\gamma}$. In the same manner we compute

$$\begin{aligned}
\rho(a)(\gamma \cdot R)(f)h &= \gamma \cdot R(\iota_{A(v)}(a)f)h = R(\delta_\gamma^{-1}(\iota_{A(v)}(a)f))h \\
&= \int_S \pi(\delta_\gamma^{-1}(\iota_{A(v)}(a)f(t)))U_t h d\beta^u(t) \\
&= \int_S \omega(\gamma)\pi(\alpha_\gamma^{-1}(af(\gamma t\gamma^{-1})))U_t h d\beta^u(t) \\
&= \pi(\alpha_\gamma^{-1}(a))\int_S \omega(\gamma)^{-1}\pi(\alpha_\gamma^{-1}(f(\gamma t\gamma^{-1})))U_t h d\beta^v(t) \\
&= \pi(\alpha_\gamma^{-1}(a))R(\delta_\gamma^{-1}(f))h = \pi(\alpha_\gamma^{-1}(a))(\gamma \cdot R)(f)h.
\end{aligned}$$

Once again using the fact that $\gamma \cdot R$ is nondegenerate, we conclude that $\rho(a) = \pi(\alpha_\gamma^{-1}(a))$. □

6.3.2 Equivalent Representations

Now it is time to explore the structure of representations induced from the stabilizers. We need to find better ways of writing them and in particular will find a couple of very nice equivalent representations. This material is at least inspired by the work done in [PSMW96] when it doesn't copy it directly. In order to proceed we need the following

Lemma 6.34 ([PSMW96, Lemma 2.1])**.** *Let G be a second countable locally compact Hausdorff groupoid. Suppose $u \in G^{(0)}$, that A is an abelian subgroup of S_u and that β is a Haar measure on A. Then the following hold.*

(a) *The formula*
$$Q(f)([\gamma]) = \int_A f(\gamma s)d\beta(s)$$
defines a surjection from $C_c(G)$ onto $C_c(G_u/A)$.

(b) *There is a non-negative, bounded, continuous function b on G_u such that for any compact set $K \subset G_u$ the support of b and KA have compact intersection and for all $\gamma \in G_u$*
$$\int_A b(\gamma s)d\beta(s) = 1. \tag{6.34}$$

(c) *There is a Radon measure σ with full support on G_u/A such that*
$$\int_G f(\gamma)d\lambda_u(\gamma) = \int_{G_u/A}\int_A f(\gamma s)d\beta(s)d\sigma([\gamma]). \tag{6.35}$$

Proof. This proof is taken (almost) verbatim from [PSMW96]. The properness of the A-action implies that G_u/A is locally compact Hausdorff and that Q takes values in $C_c(G_u/A)$. The existence of a function b' satisfying the requirements of (2) with the exception of (6.34) follows from [Bou73, Lemme 1]. Now, the rest of (2) follows by normalizing b' and then the rest of (1) follows from (2). Part (3) will follow (except for the support statement) if we can show that the equation
$$\sigma(Q(f)) = \int_G f(\gamma)d\lambda_u(\gamma)$$
yields a well-defined, positive linear functional on $C_c(G_u/A)$. But this amounts to showing that given $f \in C_c(G)$ such that
$$\int_A f(\gamma s)d\beta(s) = 0 \tag{6.36}$$

for all $\gamma \in G_u$ then $\sigma(Q(f)) = 0$. However, if (6.36) holds then for any $h \in C_c(G)$,

$$\begin{aligned}\int_A h * f(s) d\beta(s) &= \int_A \int_G h(\gamma) f(\gamma^{-1}s) d\lambda^u(\gamma) d\beta(s) \\ &= \int_G h(\gamma) \int_A f(\gamma^{-1}s) d\beta(s) d\lambda^u(\gamma) = 0\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_A h * f(s) d\beta(s) &= \int_A \int_G h(\gamma) f(\gamma^{-1}s) d\lambda^u(\gamma) d\beta(s) \\ &= \int_G \int_A h(s\gamma) f(\gamma^{-1}) d\beta(s) d\lambda^u(\gamma) \\ &= \int_G \left(\int_A (\bar{h})^*(\gamma^{-1}s) d\beta(s) \right) f(\gamma^{-1}) d\lambda^u(\gamma)\end{aligned} \tag{6.37}$$

where we replaced s by s^{-1} in the final equality and used the fact that A is abelian and hence unimodular. Now consider $K = \operatorname{supp} f \cap G_u$. By part (2) $\operatorname{supp} b \cap KA$ is compact and therefore we can use the Tietze Extension Theorem to extend b from $\operatorname{supp} b \cap KA$ to a function $d \in C_c(G)^+$. If we let $h = d^*$ then, whenever $\gamma^{-1} \in \operatorname{supp} f \cap G_u$,

$$\int_A (\bar{h})^*(\gamma^{-1}s) d\beta(s) = \int_A b(\gamma^{-1}s) d\beta(s) = 1.$$

It now follows from (6.37) that $\sigma(Q(f)) = 0$. Thus our linear functional is well defined and the Radon measure σ exists.

Next we need to show that $\operatorname{supp} \sigma = G_u/A$. Suppose O is an open neighborhood of $[\eta]$ in G_u/A. We must show $\sigma(O) > 0$. Choose $f \in C_c(G_u/A)$ such that $0 \le f \le 1$, $f([\eta]) = 1$, and $\operatorname{supp} f \subset O$. Then we have

$$\begin{aligned}\sigma(O) &\ge \int_{G_u/A} f([\gamma]) d\sigma([\gamma]) = \int_{G_u/A} \int_A f([\gamma s]) b(\gamma s) d\beta(s) d\sigma([\gamma]) \\ &= \int_G f([\gamma]) b(\gamma) d\lambda_u(\gamma).\end{aligned}$$

Now $\gamma \mapsto f([\gamma])b(\gamma)$ is a continuous function. Furthermore since $\int b(\eta s) d\beta(s) = 1$ there must be some $s \in A$ such that $b(\eta s) > 0$. However we also have $f([\eta s]) = f([\eta]) = 1 > 0$. Since $\operatorname{supp} \lambda_u = G_u$ and the integrand is continuous and nonzero on G_u, it follows that

$$\int_G f([\gamma]) b(\gamma) d\lambda_u(\gamma) > 0$$

and hence $\sigma(O) > 0$. $\square$

Remark 6.35. Suppose (A, G, α) is a separable dynamical system and that the stabilizer subgroupoid S is abelian. For all $u \in S^{(0)}$ let β^u be a Haar measure on S_u. Using Lemma 6.34 for each $u \in G^{(0)}$ there exists a Radon measure σ^u with full support on G_u/S_u such that

$$\int_G f(\gamma) d\lambda_u(\gamma) = \int_{G_u/S_u} \int_S f(\gamma s) d\beta^u(s) d\sigma^u([\gamma]).$$

For the rest of this section whenever we have (A, G, α) and S as above we will let $\sigma = \{\sigma^u\}$ be defined in this way. It is worth mentioning that if the β^u form a Haar system for S then the σ^u form a Haar system on R_Q [PSMW96, Lemma 4.2].

Lemma 6.36. *Suppose (A, G, α) is a separable dynamical system and that the stabilizer subgroupoid S is abelian. Given $u \in G^{(0)}$ let $R = \pi \rtimes U$ be a representation of $A(u) \rtimes S_u$ which acts on a separable Hilbert Space $\mathcal{H}$. Let $\mathcal{V}$ be the set of Borel functions $\phi : G_u \to \mathcal{H}$ such that* [1]

$$\phi(\gamma s) = U_s^* \phi(\gamma) \tag{6.38}$$

for all $\gamma \in G_u$ and $s \in S_u$. Define

$$\mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u) := \left\{ \phi \in \mathcal{V} : \int_{G_u/A_u} \|\phi(\gamma)\|^2 d\sigma^u([\gamma]) < \infty \right\}$$

and let $L^2_U(G_u, \mathcal{H}, \sigma^u)$ be the quotient of $\mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u)$ where we identify functions which agree λ_u-almost everywhere. If $\phi, \psi \in \mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u)$ then

$$(\phi|\psi) := \int_{G_u/S_u} (\phi(\gamma), \psi(\gamma)) d\sigma^u([\gamma]) \tag{6.39}$$

defines an inner product which makes $L^2_U(G_u, \mathcal{H}, \sigma^u)$ into a Hilbert space.

Proof. It is clear that $L^2_U(G_u, \mathcal{H}, \sigma^u)$ is at least a vector space. The usual Cauchy-Schwartz considerations will show that (6.39) is integrable and (6.38) guarantees that (6.39) is well defined on $\mathcal{L}^2_U(G_u, \mathcal{H})$. Furthermore, it is easy to see that (6.39) defines a sesqui-linear form. Let b be as in part (b) of Lemma 6.34 for S_u. Suppose

[1] Since $\mathcal{H}$ is separable we don't have to worry about the measurably considerations described in [Wil07, Appendix I.4].

$\phi \in \mathcal{L}^2_U(G_u, \mathcal{H})$ and suppose ϕ is zero λ_u-almost everywhere. Then

$$\begin{aligned}
\|\phi\|^2 &= \int_{G_u/S_u} \|\phi(\gamma)\|^2 d\sigma^u([\gamma]) \\
&= \int_{G_u/S_u} \int_S b(\gamma s)\|\phi(\gamma s)\|^2 d\beta^u(s) d\sigma^u([\gamma]) \\
&= \int_G b(\gamma)\|\phi(\gamma)\|^2 d\lambda_u(\gamma) = 0.
\end{aligned}$$

This suffices to show that (6.39) is well defined on $L^2(G_u, \mathcal{H})$. Now suppose $\|\phi\| = 0$. Then in particular $\|\phi(\gamma)\| = 0$ for all $[\gamma] \notin N$ where N is some σ^u-null set. This implies that $\|\phi(\gamma)\| = 0$ for all $\gamma \in NS_u$. However it follows from (6.35) that NS_u is λ_u-null. Hence (6.39) is positive definite on $L^2_U(G_u, \mathcal{H})$.

The last thing to show is that $L^2_U(G_u, \mathcal{H})$ is complete. This portion of the proof is inspired by [Wil07, Page 290]. Suppose ϕ_n is a Cauchy sequence. We can pass to a subsequence, relabel and assume that

$$\|\phi_{n+1} - \phi_n\| < \frac{1}{2^n}$$

for all n. We define the following extended real valued functions on G_u by

$$\begin{aligned}
z_n(\gamma) &= \sum_{i=1}^{n} \|\phi_{i+1}(\gamma) - \phi_i(\gamma)\|, \\
z(\gamma) &= \sum_{i=1}^{\infty} \|\phi_{i+1}(\gamma) - \phi_i(\gamma)\|.
\end{aligned}$$

Of course, z_n is constant on S_u orbits and factors to a Borel map on G_u/S_u. Using the triangle inequality in $L^2(G_u/S_u, \sigma^u)$ we find

$$\|z_n\| \leq \sum_{i=1}^{n} \left(\int_{G_u/S_u} \|\phi_{i+1}(\gamma) - \phi_i(\gamma)\|^2 d\sigma^u([\gamma]) \right)^{\frac{1}{2}} = \sum_{i=1}^{n} \|\phi_{i+1} - \phi_i\| \leq 1$$

Since $\|z_n\|^2 = \int_{G_u/S_u} z_n(\gamma)^2 d\sigma^u([\gamma])$ it follows from the Monotone Convergence Theorem that

$$\|z\|^2 = \int_{G_u/S_u} z(\gamma)^2 d\sigma^u([\gamma]) \leq 1.$$

Hence, there is a σ^u-null set N such that $[\gamma] \notin N$ implies that $z(\gamma) < \infty$. In particular

we can lift N to G_u and get a λ_u-null set NS_u such that $\gamma \notin NS_u$ implies

$$\sum_{i=1}^{\infty} \phi_{i+1}(\gamma) - \phi_i(\gamma) \tag{6.40}$$

is absolutely convergent. Thus (6.40) converges to some $\phi'(\gamma) \in \mathcal{H}$ for all $\gamma \notin NS$. Furthermore

$$\phi'(\gamma) = \lim_{n\to\infty} \sum_{i=1}^{n} \phi_{i+1}(\gamma) - \phi_i(\gamma) = \lim_{n\to\infty} \phi_{n+1}(\gamma) - \phi_1(\gamma)$$

Thus $\phi(\gamma) := \phi'(\gamma) - \phi_1(\gamma)$ satisfies

$$\phi(\gamma) = \lim_{n\to\infty} \phi_n(\gamma) \tag{6.41}$$

for all $\gamma \notin NS_u$. Hence $\phi_n \to \phi$ almost everywhere and ϕ is a Borel function. Now let ϕ be zero off NS_u. Then, using (6.41) and the fact that NS_u is saturated we find that

$$\phi(\gamma s) = U_s^* \phi(\gamma)$$

for all $\gamma \in G_u$ and $s \in S_u$. Next, given $\epsilon > 0$ there exists M such that $\|\phi_n - \phi_m\| < \epsilon$ for all $n, m \geq M$. If $\gamma \notin NS_u$ then

$$\|\phi(\gamma) - \phi_i(\gamma)\| = \lim_{n\to\infty} \|\phi_n(\gamma) - \phi_i(\gamma)\|.$$

Thus, if $k \geq M$, Fatou's Lemma implies that

$$\|\phi - \phi_k\|^2 \leq \liminf_{n\to\infty} \|\phi_n - \phi_k\|^2 \leq \epsilon^2$$

Furthermore we have

$$\begin{aligned}\|\phi(\gamma)\|^2 &\leq (\|\phi(\gamma) - \phi_k(\gamma)\| + \|\phi_k(\gamma)\|)^2 \\ &\leq 3\|\phi(\gamma) - \phi_k(\gamma)\|^2 + 3\|\phi_k(\gamma)\|^2\end{aligned}$$

so that

$$\int_{G_u/S_u} \|\phi(\gamma)\|^2 d\sigma^u([\gamma]) \leq 3\|\phi - \phi_k\|^2 + 3\|\phi_k\|^2 < \infty.$$

Thus $\phi \in \mathcal{L}_U^2(G_u, \mathcal{H}, \sigma^u)$, $\phi_n \to \phi$ in $L_U^2(G_u, \mathcal{H}, \sigma^u)$ and, to quote [Wil07], "this completes the proof of completeness." $\square$

The whole point of building this Hilbert space is so that we can use it to define a representation.

Lemma 6.37. *Suppose (A, G, α) is a separable dynamical system and that the stabilizer subgroupoid S is abelian. Given $u \in G^{(0)}$ let $R = \pi \rtimes U$ be a representation of $A(u) \rtimes S_u$ which acts on a separable Hilbert space $\mathcal{H}$. Then $\operatorname{Ind}_{S_u}^G R$ is equivalent to the representation T^R on $L^2_U(G_u, \mathcal{H}, \sigma^u)$ defined for $f \in \Gamma_c(G, r^*\mathcal{A})$ and $\phi \in \mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u)$ by*

$$T^R(f)\phi(\gamma) = \int_G \pi(\alpha_\gamma^{-1}(f(\gamma\eta)))\phi(\eta^{-1})d\lambda^u(\eta). \tag{6.42}$$

Proof. Recall from Theorem 6.7 that $\operatorname{Ind} R$ acts on $\mathcal{Z}^G_{S_u} \otimes_{A(u)\rtimes S_u} \mathcal{H}$ and we have $\operatorname{Ind} R(f)(z \otimes h) = f \cdot z \otimes h$ where $f \cdot z$ is given by (6.10). Now define $V : C_c(G_u, A(u)) \odot \mathcal{H} \to \mathcal{L}^2_U(G_u, \mathcal{H})$ by

$$V(z \otimes h)(\gamma) = \int_S U_s \pi(z(\gamma s))h d\beta^u(s). \tag{6.43}$$

It is straightforward to show that the integrand in (6.43) is jointly continuous in γ and s. However, this implies that it is Borel on the product space. The fact that $V(z \otimes h)$ is Borel now follows from Fubini's Theorem (for vector integration). Furthermore, given $s \in S_u$ we have

$$V(z \otimes h)(\gamma s) = \int_S U_t \pi(z(\gamma st))h d\beta^u(t) = \int_S U_{s^{-1}} U_t \pi(z(\gamma t))h d\beta^u(t) = U_s^* V(z \otimes h)(\gamma).$$

Finally, observe that $V(z \otimes h)$ is supported on the (compact) image of $\operatorname{supp} z$ in G_u/S_u so that

$$\int_{G_u/S_u} \|V(z \otimes h)(\gamma)\|^2 d\sigma^u([\gamma]) < \infty.$$

Thus $V(z \otimes h)$ maps $C_c(G_u, A(u)) \odot \mathcal{H}$ into $\mathcal{L}^2_U(G_u, \mathcal{H})$. Next, we compute

$$\begin{aligned}
(z \otimes h, w \otimes k) &= (R(\langle\langle w, z\rangle\rangle_{A(u)\rtimes S_u})h, k) \\
&= \int_S (\pi(\langle\langle w, z\rangle\rangle_{A(u)\rtimes S_u}(s))U_s h, k) d\beta^u(s) \\
&= \int_S \int_G (\pi(w(\gamma s^{-1})^* \alpha_s(z(\gamma)))U_s h, k) d\lambda_u(\gamma) d\beta^u(x) \\
&= \int_S \int_{G_u/S_u} \int_S (\pi(w(\gamma t s^{-1}))^* \pi(\alpha_s(z(\gamma t)))U_s h, k) d\beta^u(t) d\sigma^u([\gamma]) d\beta^u(s) \\
&= \int_{G_u/S_u} \int_S \int_S (\pi(w(\gamma s^{-1}))^* \pi(\alpha_{st}(z(\gamma t)))U_{st} h, k) d\beta^u(s) d\beta^u(t) d\sigma^u([\gamma])
\end{aligned}$$

where we used the fact that S_u is abelian to right translate by t. Continuing the

computation

$$\begin{aligned}(z\otimes h, w\otimes k) &= \int_{G_u/S_u}\int_S\int_S (U_sU_t\pi(z(\gamma t))h, \pi(w(\gamma s^{-1}))k)d\beta^u(s)d\beta^u(t)d\sigma^u([\gamma]) \\ &= \int_{G_u/S_u}\int_S\int_S (U_t\pi(z(\gamma t))h, U_s\pi(w(\gamma s))k)d\beta^u(s)d\beta^u(t)d\sigma^u([\gamma]) \\ &= \int_{G_u/S_u} (V(z\otimes h)(\gamma), V(w\otimes k)(\gamma))d\sigma^u([\gamma]) \\ &= (V(z\otimes h), V(w\otimes k))\end{aligned}$$

where we again used the fact that S_u is abelian and hence unimodular. Thus V is an isometry and extends to a map from $\mathcal{Z}^G_{S_u}\otimes_{A(u)\rtimes S_u}\mathcal{H}$ into $L^2_U(G_u,\mathcal{H})$. We will show that it is surjective. Suppose $\phi\in\mathcal{L}^2_U(G_u,\mathcal{H})$ is such that $(V(z\otimes h),\phi)=0$ for all $z\otimes h\in C_c(G_u,A(u))\odot\mathcal{H}$. It will suffice to show ϕ is zero λ_u-almost everywhere. We have

$$\begin{aligned}0 = (V(z\otimes h),\phi) &= \int_{G_u/S_u}(V(z\otimes h)(\gamma),\phi(\gamma))d\sigma^u([\gamma]) \qquad (6.44)\\ &= \int_{G_u/S_u}\int_S (U_s\pi(z(\gamma s))h,\phi(\gamma))d\beta^u(s)d\sigma^u([\gamma]) \\ &= \int_{G_u/S_u}\int_S (\pi(z(\gamma s))h,\phi(\gamma s))d\beta^u(s)d\sigma^u([\gamma]) \\ &= \int_G (((\pi\circ z)\otimes h)(\gamma),\phi(\gamma))d\lambda_u(\gamma).\end{aligned}$$

where $(\pi\circ z)\otimes h$ denotes the function $\gamma\mapsto\pi(z(\gamma))h$. Now, ϕ is probably not an element of $L^2(G_u,\mathcal{H})$. We get around this using the following trick. Suppose $K\subset G_u$ is compact. Let $\phi|_K$ be the function obtained by letting ϕ be zero off K, and let $g\in C_c(G_u)$ be one on K. Then

$$F([\gamma]) = \int_S g(\gamma s)d\beta^u(s)$$

defines an element of $C_c(G_u/S_u)$. We observe that

$$\begin{aligned}\int_G \|\phi|_K(\gamma)\|^2 d\lambda_u(\gamma) &\le \int_G g(\gamma)\|\phi(\gamma)\|^2 d\lambda_u(\gamma) \\ &= \int_{G_u/H_u}\|\phi(\gamma)\|^2\int_{S_u} g(\gamma s)d\beta^u(s)d\sigma^u([\gamma]) \\ &\le \|\phi\|^2\|F\|_\infty.\end{aligned}$$

Thus $\phi|_K \in L^2(G_u, \mathcal{H})$. Given $z \in C_c(G_u, A(u))$ such that $\operatorname{supp} z \subset K$ we conclude from (6.44) that

$$0 = \int_G (((\pi \circ z) \otimes h)(\gamma), \phi(\gamma)) d\lambda_u(\gamma) = ((\pi \circ z|_K) \otimes h, \phi|_K)_{L^2(K,\mathcal{H},\lambda_u)}.$$

Hence $\phi|_K$ will be zero λ_u-almost everywhere if we can show elements of the form $(\pi \circ z) \otimes h$ span a dense set in $L^2(K, \mathcal{H}, \lambda_u)$. However, we can restrict ourselves even further and show that elements of the form

$$f \otimes \pi(a)h = ((f \otimes a) \circ \pi) \otimes h$$

span a dense set, where $a \in A$, $h \in \mathcal{H}$ and $f \in C_c(K)$. Recall that $L^2(K, \mathcal{H}) \cong L^2(K) \otimes \mathcal{H}$ [Wil07, Example 2.62] and that elementary tensors span a dense set in $L^2(K, \mathcal{H})$. The result now follows quickly once we recall that $C_c(G_u)$ is dense in $L^2(G_u)$ because $\operatorname{supp} \lambda_u = G_u$ and $\pi(A(u))\mathcal{H}$ is dense in $\mathcal{H}$ because π is nondegenerate. Thus $\phi|_K$ is zero λ_u-almost everywhere for each compact set $K \subset G_u$. Since G_u is σ-compact this implies that ϕ is zero λ_u-almost everywhere.

The fact that V is a unitary implies that there is a representation T^R of $A \rtimes G$ defined by $T^R(f) = V \operatorname{Ind}_{S_u}^G R(f) V^*$. We would like to see that T^R is given by (6.42). This follows from the following computation for $f \in \Gamma_c(G, r^*\mathcal{A})$ and $\phi \in \mathcal{L}^2_U(G_u, \mathcal{H})$.

$$\begin{aligned}
T^R(f)V(z \otimes h)(\gamma) &= V \operatorname{Ind}_{S_u}^G R(f)(z \otimes h)(\gamma) = V(f \cdot z \otimes h)(\gamma) \\
&= \int_S U_s \pi(f \cdot z(\gamma s)) h d\beta^u(s) \\
&= \int_S \int_G U_s \pi(\alpha_{\gamma s}^{-1}(f(\eta)) z(\eta^{-1}\gamma s)) h d\lambda^{r(\gamma)}(\eta) d\beta^u(s) \\
&= \int_G \int_S U_s \pi(\alpha_{\gamma s}^{-1}(f(\gamma\eta)) z(\eta^{-1}s)) h d\beta^u(s) d\lambda^u(\eta) \\
&= \int_G \int_S \pi(\alpha_\gamma^{-1}(f(\gamma\eta))) U_s \pi(z(\eta^{-1}s)) h d\beta^u(s) d\lambda^u(\eta) \\
&= \int_G \pi(\alpha_\gamma^{-1}(f(\gamma\eta))) V(z \otimes h)(\eta^{-1}) d\lambda^u(\eta).
\end{aligned}$$

□

This representation is sort of "halfway" to where we want to be. In particular we would like to work with a more standard Hilbert space. The following observation will be crucial for this.

Remark 6.38. Suppose G is a second countable groupoid and fix $u \in G^{(0)}$. The fact that G_u is second countable implies that we can find a Borel cross section $c : G_u/S_u \to G_u$ for the quotient map [Arv76, Theorem 3.4.1]. Furthermore we can then define a

Borel map $\delta : G_u \to S_u$ such that $\gamma = c([\gamma])\delta(\gamma)$. We will make use of these maps in what follows. One of the key properties about δ that we will need is that

$$\delta(\gamma s) = c([\gamma s])^{-1}(\gamma s) = c([\gamma])^{-1}\gamma s = \delta(\gamma)s.$$

Using the Borel cross section δ we can transform T^R into a representation which acts on $L^2(G_u/S_u, \mathcal{H}, \sigma^u)$.

Proposition 6.39. *Suppose (A, G, α) is a separable dynamical system and that the stabilizer subgroupoid S is abelian. Given $u \in G^{(0)}$ let $R = \pi \rtimes U$ be a separable representation of $A(u) \rtimes S_u$ which acts on $\mathcal{H}$. Then T^R, and hence $\operatorname{Ind}_{S_u}^G R$, is equivalent to the representation N^R given on $L^2(G_u/S_u, \mathcal{H}, \sigma^u)$ by*

$$N^R(f)(\phi)([\gamma]) = \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(\eta)))U^*_{\delta(\eta^{-1}\gamma)}\phi([\eta^{-1}\gamma])d\lambda^{r(\gamma)}(\eta). \tag{6.45}$$

Proof. We start by defining a map W on $L^2_U(G_u, \mathcal{H}, \sigma^u)$ by $W(\phi)([\gamma]) = \phi(c([\gamma]))$. Then

$$\begin{aligned}
(W(\phi), W(\psi)) &= \int_{G_u/S_u} (W(\phi)([\gamma]), W(\psi)([\gamma]))d\sigma^u([\gamma]) \\
&= \int_{G_u/S_u} (\phi(c([\gamma])), \psi(c([\gamma])))d\sigma^u([\gamma]) \\
&= \int_{G_u/S_u} (U_{\delta(\gamma)}\phi(\gamma), U_{\delta(\gamma)}\psi(\gamma))d\sigma^u([\gamma]) \\
&= (\phi, \psi).
\end{aligned}$$

Thus W maps into $L^2(G_u/S_u, \mathcal{H}, \sigma^u)$ and is in fact isometric. Furthermore we can define an inverse by $W^{-1}(\phi)(\gamma) := U_{\delta(\gamma)}\phi([\gamma])$ so that W is actually a unitary map. We now define a representation on $A \rtimes G$ by $N^R(f) = WT^R(f)W^*$ and we show that N^R has the desired form by computing, for $f \in \Gamma_c(G, r^*\mathcal{A})$

$$\begin{aligned}
N^R(f)W\phi([\gamma]) &= WT(f)\phi([\gamma]) = T(f)\phi(c([\gamma])) \\
&= \int_G \pi(\alpha^{-1}_{c([\gamma])}(f(c([\gamma])\eta)))\phi(\eta^{-1})d\lambda^u(\eta) \\
&= \int_G \pi(\alpha^{-1}_{c([\gamma])}(f(\gamma\eta)))\phi(\eta^{-1}\delta(\gamma)^{-1})d\lambda^u(\eta) \\
&= \int_G \pi(\alpha_{\delta(\gamma)\gamma^{-1}}(f(\gamma\eta)))\phi(c([\eta^{-1}])\delta(\eta^{-1})\delta(\gamma)^{-1})d\lambda^u(\eta) \\
&= \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(\gamma\eta)))U^*_{\delta(\gamma)}U_{\delta(\gamma)}U^*_{\delta(\eta^{-1})}\phi(c([\eta^{-1}]))d\lambda_u(\eta)
\end{aligned}$$

$$
\begin{aligned}
&= \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(\gamma\eta)))U^*_{\delta(\eta^{-1})}W\phi([\eta^{-1}])d\lambda^u(\eta) \\
&= \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(\eta)))U_{\delta(\eta^{-1}\gamma)}W\phi([\eta^{-1}\gamma])d\lambda^{r(\gamma)}(\eta). \qquad \square
\end{aligned}
$$

Remark 6.40. Before we state the next proposition we need to use some more measure theoretic trickery. Observe that the range map r factors to a continuous bijection $\tilde{r}$ between G_u/S_u and $G\cdot u$. Since G_u/S_u and $G\cdot u$ are both second countable locally compact Hausdorff spaces, we cite Souslin's Theorem [Arv76, Theorem 3.2.3] to conclude that $\tilde{r}$ is a Borel isomorphism. We use $\tilde{r}$ to push the measure σ^u forward to a measure σ^u_* on $G\cdot u$. It is clear that by identifying $L^2(G_u/S_u,\mathcal{H},\sigma^u)$ and $L^2(G\cdot u,\mathcal{H},\sigma^u_*)$ via $\tilde{r}$ we can view N^R as a representation on the latter space. It's easy to see that in this case its action is given by

$$
N^R(f)(\phi)(\gamma\cdot u) = \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(\eta)))U^*_{\delta(\eta^{-1}\gamma)}\phi(\eta^{-1}\gamma\cdot u)d\lambda^{r(\gamma)}(\eta).
$$

Since this identification is fairly natural, we won't make too much of a fuss about it.

The reason we went through the effort to build N^R is that, as the next lemma demonstrates, it interfaces nicely with the multiplication representation of $C^b(G\cdot u)$ on $L^2(G\cdot u,\mathcal{H})$. We will be able to take advantage of this later on.

Lemma 6.41. *Suppose (A,G,α) is a separable dynamical system and that the stabilizer subgroupoid S is abelian. Let $u\in G^{(0)}$ and $R=\pi\rtimes U$ be a representation of $A(u)\rtimes S_u$. Consider the representation of $C_0(G^{(0)})$ on $L^2(G\cdot u,\mathcal{H},\sigma^u_*)$ defined via*

$$
N^u(f)\phi(v) = f(v)\phi(v).
$$

Furthermore, given $f\in C_0(G^{(0)})$ and $g\in\Gamma_c(G,r^*\mathcal{A})$ define $f\cdot g(\gamma) := f(r(\gamma))g(\gamma)$. Then $N^u(f)N^R(G) = N^R(f\cdot g)$ for all $f\in C_0(G^{(0)})$ and $g\in\Gamma_c(G,r^*\mathcal{A})$.

Proof. We discussed σ^u_* and how to view N^R as acting on $L^2(G\cdot u,\mathcal{H},\sigma^u_*)$ in Remark 6.40. The representation N^u is nothing more than the restriction map from $C_0(G^{(0)})$ to $C^b(G\cdot u)$ composed with the usual multiplication representation of $C^b(G\cdot u)$ on $L^2(G\cdot u,\mathcal{H})$. It is also easy to see that if f and g are as above then $f\cdot g\in\Gamma_c(G,r^*\mathcal{A})$. For the last statement we just compute

$$
\begin{aligned}
N^u(f)N^R(g)\phi(\gamma\cdot u) &= f(r(\gamma))N^R(g)\phi(\gamma\cdot u) \\
&= \int_G f(r(\gamma))U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(g(\eta)))U^*_{\delta(\eta^{-1}\gamma)}\phi(\eta^{-1}\gamma\cdot u)d\lambda^{r(\gamma)}(\eta) \\
&= \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f\cdot g(\eta)))U^*_{\delta(\eta^{-1}\gamma)}\phi(\eta^{-1}\gamma\cdot u)d\lambda^{r(\gamma)}(\eta)
\end{aligned}
$$

$$= N^R(f \cdot g)\phi(\gamma \cdot u). \qquad \square$$

The final technical aspect we need to deal with before proving something more interesting is to demonstrate the relationship between ω and the σ^u.

Lemma 6.42. *Suppose G is a second countable locally compact Hausdorff groupoid and that the stabilizer groupoid S is abelian and varies continuously. Given $u \in G^{(0)}$ and $\gamma \in G_u$ we have, letting $v = \gamma \cdot u$ and ϕ be a Borel function on G_v/S_v,*

$$\int_{G_v/S_v} \omega(\gamma)\phi([\eta\gamma])d\sigma^v([\eta]) = \int_{G_u/S_u} \phi([\eta])d\sigma^u([\eta]). \tag{6.46}$$

Proof. Since σ^u and σ^v are Radon measures, it suffices to verify (6.46) for $f \in C_c(G_u/S_u)$. Using the fact that $\eta s\gamma = \eta\gamma(\gamma^{-1}s\gamma)$ is not difficult to show that the map $[\eta] \mapsto [\eta\gamma]$ defines a homeomorphism from G_v/S_v onto G_u/S_u. Hence $[\eta] \mapsto f([\eta\gamma])$ is continuous and compactly supported. We let b be as in Lemma 6.34 and compute

$$\begin{aligned}
\int_{G_u/S_u} f([\eta])d\sigma^u([\eta]) &= \int_{G_u/S_u}\int_S f([\eta s])b(\eta s)d\beta^u(s)d\sigma^u([\eta]) \\
&= \int_G f([\eta])b(\eta)d\lambda_u(\eta) = \int_G f([\eta\gamma])b(\eta\gamma)d\lambda_v(\eta) \\
&= \int_{G_v/S_v}\int_S f([\eta s\gamma])b(\eta s\gamma)d\beta^v(s)d\sigma^v([\eta]) \\
&= \int_{G_v/S_v}\int_S \omega(\gamma)f([\eta\gamma s])b(\eta\gamma s)d\beta^u(s)d\sigma^v([\eta]) \\
&= \int_{G_v/S_v} \omega(\gamma)f([\eta\gamma])d\sigma^v([\eta]). \qquad \square
\end{aligned}$$

We can now prove the following proposition, which tells us that the equivalence classes on S induced by Φ are exactly the orbits of the G action.

Proposition 6.43. *Suppose (A, G, α) is a separable dynamical system and that the isotropy subgroupoid S is abelian and continuously varying. Let $u \in G^{(0)}$ and R be an irreducible representation of $A(u) \rtimes S_u$ on a separable Hilbert space $\mathcal{H}$. Then $\Phi(R)$ is equivalent to $\Phi(\gamma \cdot R)$ for all $\gamma \in G_u$. Furthermore if $G^{(0)}/G$ is T_0 and L and R are irreducible representations of $A(u) \rtimes S_u$ and $A(v) \rtimes S_v$, respectively, then $\Phi(L)$ is equivalent to $\Phi(R)$ if and only if there exists $\gamma \in G$ such that $\gamma \cdot L$ is equivalent to R.*

Proof. Let $v = \gamma \cdot u$. Suppose $R = \pi \rtimes U$ and $\gamma \cdot R = \rho \rtimes V$ as in Proposition 6.33. Recall that $\Phi(R) = \operatorname{Ind}_{S_u}^G R$ is equivalent to the representation T^R on $L^2_U(G_u, \mathcal{H}, \sigma^u)$ and $\Phi(\gamma \cdot R) = \operatorname{Ind}_{S_v}^G \gamma \cdot R$ is equivalent to the representation $T^{\gamma \cdot R}$ on $L^2_V(G_v, \mathcal{H}, \sigma^v)$.

We define a map W on $\mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u)$ by

$$W(\phi)(\eta) = \omega(\gamma)^{\frac{1}{2}} f(\eta\gamma) \quad \text{for all } \eta \in G_v.$$

Clearly $W(\phi)$ is Borel, and we compute for $s \in S_v$

$$\begin{aligned} W(\phi)(\eta s) &= \omega(\gamma)^{\frac{1}{2}} \phi(\eta s \gamma) = \omega(\gamma)^{\frac{1}{2}} \phi(\eta\gamma(\gamma^{-1} s \gamma)) \\ &= \omega(\gamma)^{\frac{1}{2}} U^*_{\gamma^{-1} s \gamma} \phi(\eta\gamma) = V_s^* W(\phi)(\eta). \end{aligned}$$

Furthermore, we use Lemma 6.42 to conclude that

$$\begin{aligned} (W(\phi), W(\psi)) &= \int_{G_v/S_v} (W(\phi)(\eta), W(\psi)(\eta)) d\sigma^v([\eta]) \\ &= \int_{G_v/S_v} \omega(\gamma)(\phi(\eta\gamma), \psi(\eta\gamma)) d\sigma^v([\eta]) \\ &= \int_{G_u/S_u} (\phi(\eta), \psi(\eta)) d\sigma^u([\eta]) = (\phi, \psi). \end{aligned}$$

This calculation proves two things. First, that $W(\phi)$ is in $\mathcal{L}^2_V(G_v, \mathcal{H}, \sigma^v)$ and, second, that W is isometric. Since W has an obvious inverse it must be a unitary map.

Next we show W intertwines T^R and $T^{\gamma \cdot R}$. We see for $f \in \Gamma_c(G, r^*\mathcal{A})$ that

$$\begin{aligned} WT^R(f)\phi(\eta) &= \omega(\gamma)^{\frac{1}{2}} T^R(f)\phi(\eta\gamma) \\ &= \int_G \omega(\gamma)^{\frac{1}{2}} \pi(\alpha^{-1}_{\eta\gamma}(f(\eta\gamma\zeta)))\phi(\zeta^{-1}) d\lambda^u(\zeta) \\ &= \int_G \omega(\gamma)^{\frac{1}{2}} \pi(\alpha^{-1}_{\gamma}(\alpha^{-1}_{\eta}(f(\eta\zeta))))\phi(\zeta^{-1}\gamma) d\lambda^v(\zeta) \\ &= \int_G \rho(\alpha^{-1}_{\eta}(f(\eta\zeta))) W\phi(\zeta^{-1}) d\lambda^v(\zeta) \\ &= T^{\gamma \cdot R}(f) W\phi(\eta). \end{aligned}$$

Moving on, suppose $G^{(0)}/G$ is T_0 and that we are given two irreducible representations L and R of $A(u) \rtimes S_u$ and $A(v) \rtimes S_v$, respectively. Suppose $\Phi(R)$ is equivalent to $\Phi(L)$. Proposition 6.39 implies that N^L is equivalent to N^R. Let U be the intertwining unitary and let N^u and N^v be as in Lemma 6.41. Then, given $f \in C_0(G^{(0)})$ and $g \in \Gamma_c(G, r^*\mathcal{A})$, we have

$$\begin{aligned} UN^v(f)N^R(g)h &= UN^R(f \cdot g)h = N^L(f \cdot g)Uh \\ &= N^u(f)N^L(g)Uh = N^u(f)UN^R(g)h. \end{aligned}$$

Since N^R is nondegenerate this implies that N^v is unitarily equivalent to N^u. However, if $G \cdot u \cap G \cdot v = \emptyset$ then [Wil81, Lemma 4.15] says that N^v and N^u can have no equivalent subrepresentations. Therefore we must have $G \cdot u = G \cdot v$. So let $\gamma \in G$ be such that $\gamma \cdot u = v$. Then $\gamma \cdot L$ and R are both irreducible representation of $A(v) \rtimes S_v$ and we assumed that that $\Phi(R)$ is equivalent to $\Phi(L)$ which is in turn equivalent to $\Phi(\gamma \cdot L)$ by the above. It now follows from Proposition 6.24 that $\gamma \cdot L$ is equivalent to R. $\square$

6.3.3 Restricting Representations

Now that we know which representations have the same image under Φ it is time to try and show that Φ is open. The key construction is a restriction process from $A \rtimes G$ to $A \rtimes S$. This is defined using the following map.

Proposition 6.44. *Suppose (A, G, α) is a separable dynamical system and the stabilizer subgroupoid S is abelian and continuously varying. Then there is a nondegenerate homomorphism $M : A \rtimes S \to M(A \rtimes G)$ such that*

$$M(f)g(\gamma) = \int_S f(s)\alpha_s(g(s^{-1}\gamma))d\beta^{r(\gamma)}(s) \tag{6.47}$$

for $f \in \Gamma_c(S, p^*\mathcal{A})$ and $g \in \Gamma_c(G, r^*\mathcal{A})$.

Proof. Since $M(f)g$ is basically given by convolution, it is straightforward to show that $M(f)g \in \Gamma_c(G, r^*\mathcal{A})$ and we will not detail a proof here. Instead, we show that M_f is adjointable and $A \rtimes G$-linear on $\Gamma_c(G, r^*\mathcal{A})$. First

$$\begin{aligned}
M(f)(g * h)(\gamma) &= \int_S \int_G f(s)\alpha_s(g(\eta)\alpha_\eta(h(\eta^{-1}s^{-1}\gamma)))d\lambda^{r(\gamma)}(\eta)d\beta^{r(\gamma)}(s) \\
&= \int_S \int_G f(s)\alpha_s(g(\eta))\alpha_{s\eta}(h(\eta^{-1}s^{-1}\gamma))d\lambda^{r(\gamma)}(\eta)d\beta^{r(\gamma)}(s) \\
&= \int_S \int_G f(s)\alpha_s(g(s^{-1}\eta))\alpha_\eta(h(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta)d\beta^{r(\gamma)}(s) \\
&= \int_G M(f)g(\eta)\alpha_\eta(h(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta) \\
&= (M(f)g) * h(\gamma).
\end{aligned}$$

Next we compute

$$(M(f)g)^* * h(\gamma) = \int_G \alpha_\eta(M(f)g(\eta^{-1})^* h(\eta^{-1}\gamma))d\lambda^{r(\gamma)}(\eta)$$

$$
\begin{aligned}
&= \int_G \int_S \alpha_\eta(\alpha_s(g(s^{-1}\eta^{-1})^*)f(s)^*h(\eta^{-1}\gamma))d\beta^{s(\eta)}(s)d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \int_S \omega(\eta^{-1})\alpha_\eta(\alpha_{\eta^{-1}s\eta}(g(\eta^{-1}s^{-1})^*)f(\eta^{-1}s\eta)^*h(\eta^{-1}\gamma))d\beta^{r(\gamma)}(s)d\lambda^{r(\gamma)}(\eta) \\
&= \int_S \int_G \omega(\eta^{-1}s)\alpha_{s^{-1}\eta}(\alpha_{\eta^{-1}s\eta}(g(\eta^{-1})^*)f(\eta^{-1}s\eta)^*h(\eta^{-1}s\gamma))d\lambda^{r(\gamma)}(\eta)d\beta^{r(\gamma)}(s) \\
&= \int_G \int_S \omega(\eta^{-1})\alpha_\eta(g(\eta^{-1})^*)\alpha_{s^{-1}\eta}(f(\eta^{-1}s\eta)^*h(\eta^{-1}s\gamma))d\beta^{r(\gamma)}(\eta)d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \int_S g^*(\eta)\alpha_{\eta s^{-1}}(f(s)^*h(s\eta^{-1}\gamma))d\beta^{s(\eta)}(s)d\lambda^{r(\gamma)}(\eta) \\
&= \int_G \int_S g^*(\eta)\alpha_\eta(f^*(s)\alpha_s(h(s^{-1}\eta^{-1}\gamma)))d\beta^{s(\eta)}(s)d\lambda^{r(\gamma)}(\eta) \\
&= g^* * (M(f^*)h)(\gamma)
\end{aligned}
$$

where we have used the fact that ω is S-invariant and S_u is unimodular. Finally we show that M preserves convolution on $\Gamma_c(S, p^*\mathcal{A})$ by calculating

$$
\begin{aligned}
M(f*g)h(\gamma) &= \int_S \int_S f(t)\alpha_t(g(t^{-1}s))\alpha_s(h(s^{-1}\gamma))d\beta^{r(\gamma)}(t)d\beta^{r(\gamma)}(s) \\
&= \int_S f(t)\alpha_t(g(s))\alpha_{ts}(h(s^{-1}t^{-1}\gamma))d\beta^{r(\gamma)}(s)d\beta^{r(\gamma)}(t) \\
&= \int_S f(t)\alpha_t(g(s)\alpha_s(s^{-1}t^{-1}\gamma))d\beta^{r(\gamma)}(s)d\beta^{r(\gamma)}(t) \\
&= M(f)M(g)h(\gamma).
\end{aligned}
$$

Moving on, we show that elements of the form $M(f)g$ are dense in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the inductive limit topology. As in Section 6.1, this argument will be a lengthy one. Fix $\epsilon > 0$ and suppose $g \in \Gamma_c(G, r^*\mathcal{A})$. Let $K = r(\operatorname{supp} g)$ and choose some fixed open neighborhood U of K in S. We make the following claim.

Claim. There is a relatively compact open neighborhood O of K in S such that $O \subset U$ and for all $\gamma \in G$ and $s \in O$

$$\|\alpha_s(g(s^{-1}\gamma)) - g(\gamma)\| < \epsilon/2. \tag{6.48}$$

Proof of Claim. Suppose not. Then for every relatively compact neighborhood $W \subset U$ of K there exists $\gamma_W \in G$ and $s_W \in W$ such that

$$\|\alpha_{s_W}(g(s_W^{-1}\gamma_W)) - g(\gamma_W)\| \geq \epsilon/2. \tag{6.49}$$

When we order W by reverse inclusion the sets $\{\gamma_W\}$ and $\{s_W\}$ form nets in G and

S respectively. In order for (6.49) to hold we must have either $s_W^{-1}\gamma_W \in \operatorname{supp} g$ or $\gamma_W \in \operatorname{supp} g$ for each W. In either case we have $r(\gamma_W) \in K$ and, since W is a neighborhood of K, $\gamma_W \in W \operatorname{supp} g \subset \overline{U} \operatorname{supp} g$. Furthermore, $s_W \in W \subset \overline{U}$ for all W. Since $\overline{U}$ and $\overline{U} \operatorname{supp} g$ are compact, we can pass to a subnet, twice, relabel, and find $s \in S$ and $\gamma \in G$ such that $s_W \to s$ and $\gamma_W \to \gamma$. However, since s_W is eventually in every neighborhood of K we must have $s \in K \subset G^{(0)}$. This implies that $s_W^{-1}\gamma_W \to \gamma_W$. However, using the continuity of the action, this contradicts (6.49). □

Let O be the open set from above and let $f \in C_c(S)^+$ such that $\operatorname{supp} f \subset O$ and f is one on K. Then the function

$$u \mapsto c(u) := \int_S f(s) d\beta^u(s)$$

is continuous and nonzero on K. In particular, the function $1/c$ is continuous on K so that we may use the Tietze Extension Theorem to find $\tilde{c} \in C_c(G^{(0)})$ which extends $1/c$ off K. Then we can replace f by $(\tilde{c} \circ p)f$ and assume without loss of generality that

$$\int_S f(s)\beta^u(s) = 1$$

for all $u \in K$. Next, let $\{a_l\}$ be an approximate identity for A. We make the following claim.

Claim. There exists l_0 such that

$$\|a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma)) - \alpha_s(g(s^{-1}\gamma))\| < \epsilon/2 \tag{6.50}$$

for all $s \in \operatorname{supp} f$ and $\gamma \in G$.

Proof of Claim. Suppose not. Then for each l there exists $\gamma_l \in G$ and $s_l \in \operatorname{supp} f$ such that

$$\|a_l(r(\gamma_l))\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - \alpha_{s_l}(g(s_l^{-1}\gamma_l))\| \geq \epsilon/2. \tag{6.51}$$

However, in order for (6.51) to hold we must have $s_l^{-1}\gamma_l \in \operatorname{supp} g$ for all l. But then $\gamma_l \in (\operatorname{supp} f)^{-1} \operatorname{supp} g$. Since both this set and $\operatorname{supp} f$ are compact we can pass to two subnets, relabel, and find $\gamma \in G$ and $s \in S$ such that $\gamma_l \to \gamma$ and $s_l \to s$. However we now have $\alpha_{s_l}(g(s_l^{-1}\gamma_l)) \to \alpha_s(g(s^{-1}\gamma))$. Choose $b \in A$ such that $b(r(\gamma)) = \alpha_s(g(s^{-1}\gamma))$. Then $a_l b \to b$ and $b(r(\gamma_l)) \to b(r(\gamma))$. Since $\alpha_{s_l}(g(s_l^{-1}\gamma_l)) \to b(r(\gamma))$, we must have

$$\|\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - b(r(\gamma_l))\| \to 0.$$

Putting everything together, it follows that, eventually,

$$\|a_l(r(\gamma_l))\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - \alpha_{s_l}(g(s_l^{-1}\gamma_l))\| \leq \|a_l(r(\gamma_l))\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - a_l(r(\gamma_l))b(r(\gamma_l))\|$$

$$
\begin{aligned}
&+ \|a_l(r(\gamma_l))b(r(\gamma_l)) + b(r(\gamma_l))\| \\
&+ \|b(r(\gamma_l)) - \alpha_{s_l}(g(s_l^{-1}\gamma_l))\| \\
\leq &2\|\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - b(r(\gamma_l))\| + \|a_l b - b\| \\
< &\epsilon/2
\end{aligned}
$$

and this contradicts (6.51). □

Now consider $f \otimes a_{l_0} \in \Gamma_c(S, p^*\mathcal{A})$. First observe that $\operatorname{supp} f \otimes a_{l_0} \subset U$ and that U was chosen independently of ϵ. Next, given $\gamma \in G$ if $r(\gamma) \notin K$ then $g(s\gamma) = 0$ for all $s \in S_{r(\gamma)}$ so that in particular

$$
M(f \otimes a_{l_0})g(\gamma) - g(\gamma) = \int_S f(s)a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma))d\beta^{r(\gamma)}(s) = 0.
$$

If $r(\gamma) \in K$ then

$$
\begin{aligned}
&\|M(f \otimes a_{l_0})g(\gamma) - g(\gamma)\| \\
&\qquad = \left\| \int_S f(s)a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma))d\beta^{r(\gamma)}(s) - \int_S f(s)d\beta^{r(\gamma)}(s)g(\gamma) \right\| \\
&\qquad \leq \int_S f(s)\|a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma)) - g(\gamma)\|d\beta^{r(\gamma)}(s) \\
&\qquad \leq \int_S f(s)\|a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma)) - \alpha_s(g(s^{-1}\gamma))\|d\beta^{r(\gamma)}(s) \\
&\qquad\quad + \int_S f(s)\|\alpha_s(g(s^{-1}\gamma)) - g(\gamma)\|d\beta^{r(\gamma)}(s) \\
&\qquad\quad < \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}
$$

Hence $\|M(f \otimes a_{l_0})g - g\|_\infty < \epsilon$. This suffices to show that elements of the form $M(f)g$ are dense in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the inductive limit topology.

Next, we want to show that $M(f)$ is bounded so that it extends to a multiplier on $A \rtimes G$. Let ρ be a state on $A \rtimes G$ and define an inner product on $\Gamma_c(G, r^*\mathcal{A})$ via

$$
(f, g)_\rho = \rho(\langle f, g \rangle)
$$

where we give $A \rtimes G$ its usual inner-product as an $A \rtimes G$-module. This is clearly sesqui-linear and is positive because states are positive. Let $\mathcal{H}_\rho$ be the Hilbert space completion of $\Gamma_c(G, r^*\mathcal{A})$ with respect to this pre-inner product. Just as in the proof of Proposition 6.8, we would like to show that we can apply Theorem 3.119 when $\mathcal{H}_0$ is the image of $\Gamma_c(G, r^*\mathcal{A})$ in $\mathcal{H}_\rho$. Define π on $\mathcal{H}_0$ by

$$
\pi(f)g = M(f)g
$$

for $f \in \Gamma_c(S, p^*\mathcal{A})$ and $g \in \Gamma_c(G, r^*\mathcal{A})$. If $(g,h)_\rho = 0$ for all $h \in \Gamma_c(G, r^*\mathcal{A})$ then

$$(\pi(f)g, h)_\rho = \rho((M(f)g)^* * h) = \rho(g^* * M(f^*)h) = (g, M(f^*)h)_\rho = 0.$$

Thus, π is well defined and it is clear that π is a homomorphism from $\Gamma_c(S, p^*\mathcal{A})$ to the algebra of operators on $\mathcal{H}_0$. Next, elements of the form $M(f)g$ are dense in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the inductive limit topology and therefore with respect to the norm topology as well. It follows that elements of the form $\pi(f)g$ are dense in $\mathcal{H}_\rho$. Fix $g, h \in \Gamma_c(G, r^*\mathcal{A})$. We would like to see that $f \mapsto (\pi(f)g, h)_\rho$ is continuous with respect to the inductive limit topology. It suffices to see that the map $f \mapsto M(f)g$ is continuous with respect to the inductive limit topology. Suppose $f_i \to f$ uniformly and $\operatorname{supp} f_i \subset K$ for some compact set K. Then

$$\begin{aligned}
\|M(f_i)g(s) - M(f)g(s)\| &\leq \int_S \|f_i(s) - f(s)\|\|\alpha_s(g(s^{-1}\gamma))\| d\beta^{r(\gamma)}(s) \\
&\leq \int_S \|f_i(s) - f(s)\|\|g\|_\infty d\beta^{r(\gamma)}(s) \\
&\leq \|f_i - f\|_\infty \|g\|_\infty \beta^{r(\gamma)}(K)
\end{aligned}$$

Since $\{\beta^u(K)\}$ is bounded this shows that $M(f_i)g \to M(f_i)g$ uniformly. Furthermore it is easy to see that $\operatorname{supp} M(f_i)g \subset K \operatorname{supp} g$ so that $M(f_i)g \to M(f)g$ with respect to the inductive limit topology. Finally, the fact that $(\pi(f)g, h)_\rho = (g, \pi(f^*)h)_\rho$ follows immediately from the fact that $(M(f)g)^* * h = g^* * (M(f^*)h)$. Thus, it follows from Theorem 3.119 that π extends to a representation of $A \rtimes G$. In particular we have

$$\rho(\langle M(f)g, M(f)g\rangle) = (\pi(f)g, \pi(f)g)_\rho \leq \|f\|^2 (g,g)_\rho \leq \|f\|^2 \|g\|^2.$$

By choosing ρ so that $\rho(\langle M(f)g, M(f)g\rangle) = \|M(f)g\|^2$ we conclude that $\|M(f)g\| \leq \|f\|\|g\|$. Thus $M(f)$ is bounded and is adjointable with adjoint $M(f^*)$ on a dense subspace and therefore extends to a multiplier on $A \rtimes G$. Furthermore $\|M(f)\| \leq \|f\|$ so that M also extends to all of $A \rtimes S$. We have shown that M is a homomorphism on a dense subspace so it is a homomorphism everywhere. Finally, the fact that M is nondegenerate follows from the fact that elements of the form $M(f)g$ are dense in the inductive limit topology. $\square$

The point is that nondegenerate maps into multiplier algebras yield continuous restriction processes through the usual general nonsense.

Corollary 6.45. *Suppose (A, G, α) is a separable dynamical system and that the stabilizer subgroupoid S is abelian and continuously varying. Then there exists a restriction map $\operatorname{Res}_M : \mathcal{I}(A \rtimes G) \to \mathcal{I}(A \rtimes S)$ such that Res_M is continuous and is characterized by $\operatorname{Res}_M(\ker \pi) = \ker \overline{\pi} \circ M$ for all representations π of $A \rtimes G$.*

Proof. We proved in Proposition 6.44 that $M : A \rtimes S \to M(A \rtimes G)$ is a nondegenerate homomorphism. Therefore it follows as in the latter half of [RW98, Page 61] that M defines the required continuous restriction map. $\square$

One might be tempted into thinking that restriction and induction are dual, or inverse, in some sense. The next lemma shows that this is not the case.

Lemma 6.46. *Suppose (A, G, α) is a separable dynamical system and that the stabilizer subgroupoid S is abelian and continuously varying. Then given $u \in G^{(0)}$ and an irreducible representation $R = \pi \rtimes U$ of $A(u) \rtimes S_u$ we have*

$$\mathrm{Res}_M \ker \mathrm{Ind}_{S_u}^G R = \bigcap_{\gamma \in G_u} \ker(\gamma \cdot R). \tag{6.52}$$

Proof. We know from Proposition 6.39 that $\mathrm{Ind}_{S_u}^G R$ is equivalent to N^R acting on $L^2(G_u/S_u, \mathcal{H}, \sigma^u)$ via (6.45). Let $Q = \overline{N^R} \circ M$ so that

$$\mathrm{Res}_M \ker \mathrm{Ind}_{S_u}^G R = \mathrm{Res}_M \ker N^R = \ker Q.$$

Given $\gamma \in G_u$ recall that we can decompose $\gamma \cdot R$ as $\pi^\gamma \rtimes U^\gamma$ where π^γ and U^γ are given in Proposition 6.33. Furthermore, we will leave it to the reader to keep track of when we are treating R as a representation of $A(u) \rtimes S_u$ or of $A \rtimes S$.

Given $f \in A \rtimes S$ it is straightforward to show that the collection $\{c([\gamma]) \cdot R(f)\}$ is a Borel field of operators on the trivial bundle $G_u/S_u \times \mathcal{H}$. Use Proposition 3.93 to form the direct integral representation $\int_{G_u/S_u}^{\oplus} c([\gamma]) \cdot R \, d\sigma^u([\gamma])$. We can then compute for $f \in \Gamma_c(S, p^*\mathcal{A})$, $g \in \Gamma_c(G, r^*\mathcal{A})$ and $\phi \in \mathcal{L}^2(G_u/S_u, \mathcal{H}, \sigma^u)$ that

$$\begin{aligned}
&Q(f)N^R(g)\phi([\gamma]) = N^R(M(f)g)\phi([\gamma]) \\
&\quad = \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(M(f)g(\eta)))U^*_{\delta(\eta^{-1}\gamma)}\phi([\eta^{-1}\gamma])d\lambda^{r(\gamma)}(\eta) \\
&\quad = \int_G \int_S U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(s)\alpha_s(g(s^{-1}\eta))))U^*_{\delta(\eta^{-1}\gamma)}\phi([\eta^{-1}\gamma])d\beta^{r(\gamma)}(s)d\lambda^{r(\gamma)}(\eta) \\
&\quad = \int_S \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(s)\alpha_s(g(\eta))))U^*_{\delta(\eta^{-1}s^{-1}\gamma)}\phi([\eta^{-1}s^{-1}\gamma])d\lambda^{r(\gamma)}(\eta)d\beta^{r(\gamma)}(s) \\
&\quad = \int_S \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(s)\alpha_s(g(\eta))))U^*_{\gamma^{-1}s^{-1}\gamma}U^*_{\delta(\eta^{-1}\gamma)}\phi([\eta^{-1}\gamma])d\lambda^{r(\gamma)}(\eta)d\beta^{r(\gamma)}(s) \\
&\quad = \int_S \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(s)))U_{\gamma^{-1}s\gamma}\pi(\alpha_\gamma^{-1}(g(\eta)))U^*_{\delta(\eta^{-1}\gamma)}\phi([\eta^{-1}\gamma])d\lambda^{r(\gamma)}(\eta)d\beta^{r(\gamma)}(s) \\
&\quad = \int_S \int_G \pi(\alpha_{\delta(\gamma)\gamma^{-1}}(f(s)))U_{\delta(\gamma)\gamma^{-1}s\gamma\delta(\gamma)^{-1}}U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(g(\eta)))\dots \\
&\qquad\qquad \dots U^*_{\delta(\eta^{-1}\gamma)}\phi([\eta^{-1}\gamma])d\lambda^{r(\gamma)}(\eta)d\beta^{r(\gamma)}(s)
\end{aligned}$$

$$
\begin{aligned}
&= \int_S \pi(\alpha^{-1}_{c([\gamma])}(f(s)))U_{c([\gamma])^{-1}sc([\gamma])}N^R(g)\phi([\gamma])d\beta^{r(\gamma)}(s) \\
&= \pi^{c([\gamma])} \rtimes U^{c([\gamma])}(f)N^R(g)\phi([\gamma]) = c([\gamma]) \cdot R(f)(N^R(g)\phi([\gamma])) \\
&= \int^{\oplus}_{G_u/S_u} c([\eta]) \cdot R(f)\, d\sigma^u([\eta])N^R(g)\phi([\gamma]).
\end{aligned}
$$

Since N^R is nondegenerate, this implies that $Q = \int^{\oplus}_{G_u/S_u} c([\gamma]) \cdot R\, d\sigma^u([\gamma])$ and that

$$Q(f)\phi([\gamma]) = (c([\gamma]) \cdot R)(f)\phi([\gamma]) \tag{6.53}$$

for all $f \in A \rtimes S$ and $\phi \in \mathcal{L}^2(G_u/S_u, \mathcal{H})$. Now suppose $f \in A \rtimes S$ and $Q(f) = 0$. Let $g_i \in C_c(G_u/S_u)$ be a countable set of functions which separate points and let h_j be a countable basis for $\mathcal{H}$. Then for each g_i and h_j we have

$$(c([\gamma]) \cdot R)(f)(g_i \otimes h_j)([\gamma]) = g_i([\gamma])(c([\gamma]) \cdot R)(f)h_j = 0 \tag{6.54}$$

for all $[\gamma] \notin N_{ij}$ where N_{ij} is a σ^u-null set. Let $N = \bigcup_{ij} N_{ij}$ and observe that given $[\gamma] \notin N$ (6.54) holds for all i and j. In particular, we can pick g_i so that $g_i([\gamma]) \neq 0$ and conclude that $(c([\gamma]) \cdot R)(f) = 0$. Thus $(c([\gamma]) \cdot R)(f) = 0$ for all $[\gamma] \notin N$. Now consider NS_u. Since N is a σ^u-null set it follows from (6.35) that NS_u is λ_u-null. We conclude that $(c([\gamma]) \cdot R)(f) = 0$ for almost every $\gamma \in G_u$.

Next, suppose $s \in S_u$ and $s \cdot R = \pi^s \rtimes U^s$. We have $U^s_t = U_{s^{-1}ts} = U_t$ and $\pi^s = \pi \circ \alpha_s^{-1}$. Then for each $f \in C_c(S_u, A(u))$ we compute

$$\pi^s \rtimes U^s(f) = \int_S \pi(\alpha_s^{-1}(f(t)))U_t d\beta^u(t) = \int_S U_s^* \pi(f(s))U_s U_t d\beta^u(t) = U_s^* \pi \rtimes U(f)U_s.$$

Hence $s \cdot R$ and R are unitarily equivalent. In particular $\gamma \cdot R = c([\gamma]) \cdot (\delta(\gamma) \cdot R) \cong c([\gamma]) \cdot R$ and the previous paragraph implies that $\gamma \cdot R(f) = 0$ for λ_u-almost all γ. Since G acts continuously on $(A \rtimes S)^\wedge$, the map $\gamma \mapsto \gamma \cdot R(f)$ is continuous. Furthermore $\operatorname{supp} \lambda_u = G_u$ and $\gamma \cdot R(f) = 0$ for λ_u-almost every $\gamma \in G_u$ so that we must have $\gamma \cdot R(f) = 0$ for all $\gamma \in G_u$. Hence $\ker Q \subset \bigcap_{\gamma \in G_u} \ker(\gamma \cdot R)$. The other inclusion is obvious since if $f \in \ker(\gamma \cdot R)$ for all $\gamma \in G_u$ then for any $\phi \in L^2(G_u/S_u, \mathcal{H})$ we have

$$Q(f)\phi([\gamma]) = c([\gamma]) \cdot R(f)\phi([\gamma]) = 0$$

for all $[\gamma]$. Thus $Q(f) = 0$ and $\ker Q = \bigcap_{\gamma \in G_u} \ker(\gamma \cdot R)$. □

Remark 6.47. Since Q is a representation of a $C_0(G^{(0)})$-algebra it must have a decom-

position. We showed in the proof of Lemma 6.46 that Q decomposes as

$$Q = \int_{G_u/S_u}^{\oplus} c([\gamma]) \cdot R \, d\sigma^u([\gamma]).$$

where we view $L^2(G_u/S_u, \mathcal{H}, \sigma^u)$ as sections of the trivial bundle. Furthermore, modulo some σ-finite concerns, we could use the fact that G_u/S_u is Borel isomorphic to $G \cdot u \subset G^{(0)}$ to view this as a decomposition over $G^{(0)}$.

6.3.4 The Main Result

We now have all but one of the tools we need to prove the main result. In order to make proper use of our restriction map we need this useful technical lemma.

Lemma 6.48 ([Wil07, Lemma 8.38])**.** *Suppose A is a C^*-algebra. For each closed set $F \subset \operatorname{Prim} A$ let $I(F)$ be the corresponding ideal in $\mathcal{I}(A)$. Then a net $\{I(F_j)\}$ converges to $I(F)$ in $\mathcal{I}(A)$ if and only if given $P \in F$ there is a subnet $\{I(F_{j_k})\}$ and $P_k \in F_{j_k}$ such that $P_k \to P$ in $\operatorname{Prim} A$.*

Proof. Suppose that $I(F_j) \to I(F)$ in $\mathcal{I}(A)$ and that $P \in F$. Let U be a neighborhood of P in $\operatorname{Prim} A$ and let $J = I(\operatorname{Prim} A \setminus U)$ be the ideal corresponding to the complement of U. Then $I(F) \not\supset J$ and therefore

$$\mathcal{O}_J = \{ I \in \mathcal{I}(A) : I \not\supset J \}$$

is a neighborhood of $I(F)$. Thus there is a j_0 such that $j \geq j_0$ implies that $I(F_j) \in \mathcal{O}_J$. In particular, if $j \geq j_0$ then $U \cap F_j \neq \emptyset$. Next, if we define

$$M := \{ (U, j) : U \text{ is a neighborhood of } P \text{ and } U \cap F_j \neq \emptyset \}.$$

then M is directed by decreasing U and increasing j. Observe that $\{I(F_{U,j})\}$ is a subnet of $\{I(F_j)\}$. For each $m = (U, j) \in M$ we can pick $P_m \in F_j \cap U$. Then $\{P_m\}$ converges to P as required.

For the converse, suppose that $\{I(F_j)\}$ has the property given in the lemma and that $I(F_j) \not\to I(F)$. After passing to a subnet, and relabeling, we can assume that there is an open set $U \subset \operatorname{Prim} A$ such that $U \cap F \neq \emptyset$ and such that $F_j \cap U = \emptyset$ for all j. But if $P \in F \cap U$ then we can pass to a subnet, relabel, and find $P_j \in F_j$ such that $P_j \to P$. Then P_j must eventually be in U which is a contradiction. $\square$

We have now acquired everything we need to identify the spectrum of $A \rtimes G$.

Theorem 6.49. *Suppose (A, G, α) is a separable dynamical system and that the isotropy subgroupoid S is abelian and has continuously varying stabilizers. If $G^{(0)}/G$*

is T_0 then $\Phi : (A \rtimes S)^\wedge \to (A \rtimes G)^\wedge$ defined by $\Phi(R) = \operatorname{Ind}_S^G R$ is open and factors to a homeomorphism from $(A \rtimes S)^\wedge/G$ onto $(A \rtimes G)^\wedge$.

Proof. It follows from Proposition 6.27 that Φ is a continuous surjection and from Proposition 6.43 that Φ factors to a bijection on $(A \rtimes S)^\wedge/G$. All that remains is to show that Φ is open. We will use Proposition 1.25. Suppose $\Phi(R_i) \to \Phi(R)$ so that, almost by definition, $\ker \Phi(R_i) \to \ker \Phi(R)$. Using Corollary 6.45 we know that Res_M is continuous and therefore

$$\operatorname{Res}_M \ker \Phi(R_i) = \operatorname{Res}_M \ker \operatorname{Ind}_S^G R_i \to \operatorname{Res}_M \ker \Phi(R) = \operatorname{Res}_M \ker \operatorname{Ind}_S^G R.$$

Let $u = \sigma(R)$ and $u_i = \sigma(R_i)$ for all l where $\sigma : (A \rtimes S)^\wedge \to S^{(0)}$ is the natural map. Using the identifications made in Remark 6.29 and Lemma 6.46 we have

$$\begin{aligned} \operatorname{Res}_M \ker \operatorname{Ind}_S^G R &= \bigcap_{\gamma \in G_u} \ker(\gamma \cdot R), \quad \text{and} \\ \operatorname{Res}_M \ker \operatorname{Ind}_S^G R_i &= \bigcap_{\gamma \in G_{u_i}} \ker(\gamma \cdot R_i) \end{aligned}$$

for all i. It follows from the definition of the Jacobson topology that the closed sets associated to $\operatorname{Res}_M \ker \operatorname{Ind}_S^G R$ and $\operatorname{Res}_M \ker \operatorname{Ind}_S^G R_i$ are

$$\begin{aligned} F &= \overline{\{\ker \gamma \cdot R : \gamma \in G_u\}}, \quad \text{and} \\ F_i &= \overline{\{\ker \gamma \cdot R_i : \gamma \in G_{u_i}\}}, \end{aligned}$$

respectively. Since $\ker R \in F$ it follows from Lemma 6.48 that, after passing to a subnet and relabeling, there exists $P_i \in F_i$ such that $P_i \to \ker R$.

Let $\mathcal{U} = \{U\}$ be a neighborhood basis of $\ker R$. For each $U \in \mathcal{U}$ there exists i_0 such that $i \geq i_0$ implies that $P_i \in U$. We let

$$M := \{(U, i) : U \in \mathcal{U}, P_i \in U\}.$$

and direct M by decreasing U and increasing i. Then M is a subnet of i such that $P_{(U,i)} \in U$ for all $(U, i) \in M$. Given $(U, i) \in M$ since U is an open set containing P_i there exists $\gamma_{(U,i)} \in G_{u_i}$ such that $\ker \gamma_{(U,i)} \cdot R_i \in U$. Now, given any $U_0 \in \mathcal{U}$, choose i_0 so that $P_{i_0} \in U$ and $(U_0, i_0) \in M$. If $(U, i) \in M$ such that $(U_0, i_0) \leq (U, i)$ then $\ker \gamma_{(U,i)} \cdot R_i \in U \subset U_0$. Thus

$$\ker \gamma_{(U,i)} \cdot R_i \to \ker R.$$

However this implies that any ideal not contained in $\ker R$ is eventually not contained in $\ker \gamma_{(U,i)} \cdot R_i$. Thus, by definition, $\gamma_{(U,i)} \cdot R_i \to R$. This suffices to show that Φ is

open. □

Remark 6.50. If there is a problem with Theorem 6.49 it is that $(A \rtimes S)^\wedge$ can be just as mysterious as $(A \rtimes G)^\wedge$. As we will see, and have seen, there are times when $(A \rtimes S)^\wedge$ can be analyzed, but in general it is difficult. For instance, if A has Hausdorff spectrum (and is separable) then each fibre $A(u)$ can be identified with the compacts and in this case $A(u) \rtimes S_u$ is relatively well understood [Wil07, Section 7.3] and in particular is isomorphic to $C^*(S_u, \bar{\omega}_u)$ where $[\omega_u]$ is the Mackey obstruction for $\alpha|_{S_u}$. However, even if the stabilizers vary continuously, the collection $\{\omega_u\}$ may be poorly behaved and identifying the total space topology of $(A \rtimes S)^\wedge$ may be difficult. On the bright side, $A \rtimes S$ is a bundle product and there are times when we can say something about it. For instance, consider the scalar case. Then $(A \rtimes S)^\wedge$ becomes $\widehat{S}$ which is much simpler. Or consider the case where α is "locally unitary on the stabilizers." Then Theorem 5.58 tells us that $(A \rtimes S)^\wedge$ is a principal bundle. In particular, it is determined up to isomorphism by a cohomology class and thus $(A \rtimes G)^\wedge$ has a nice cohomological invariant.

The following corollary is immediate and interesting enough to be worth writing down. We will explore further applications of Theorem 6.49 in the next chapter.

Corollary 6.51. *Suppose (A, G, α) is a separable dynamical system, that G is a principal groupoid, and that $G^{(0)}/G$ is T_0. Then $(A \rtimes G)^\wedge$ is isomorphic to $\widehat{A}/G$.*

Proof. Since G is principal it clearly has continuously varying abelian isotropy. In fact the isotropy subgroupoid is just $G^{(0)}$. Furthermore, we have $A \rtimes G^{(0)} = A$ and the result now follows from Theorem 6.49. □

Chapter 7

Examples and Applications

In this chapter we present a number of applications of Theorem 6.49. Section 7.1 contains a strengthening of the main result in the scalar case. The scalar case is particularly interesting because we identify the spectrum of $C^*(G)$ as a quotient of the much better understood $\widehat{S}$. In Section 7.2 we apply these results to transformation groupoids and transformation groupoid algebras. This allows us to present a couple of interesting examples and as well as connect these results back to the existing theory. Finally, in Section 7.3 we give a partial analysis of when a groupoid C^*-algebra has Hausdorff spectrum.

7.1 Groupoid Algebras with Abelian Isotropy

We would like to address the concerns made in Remark 6.50 and show that, at least in the scalar case, we can come up with a much more concrete result. In particular, the topology on $\widehat{S}$ is well understood so that the following corollary to Theorem 6.49 gives us a very useful identification of the topology of $C^*(G)^\wedge$.

Corollary 7.1. *Let G be a second countable locally compact Hausdorff groupoid with a Haar system. Furthermore, suppose the stabilizer subgroupoid S is abelian and varies continuously. If $G^{(0)}/G$ is T_0 then the map $\omega \mapsto \operatorname{Ind}_S^G \omega$ is open and factors to a homeomorphism of $\widehat{S}/G$ onto $C^*(G)^\wedge$.*

Proof. If G is as above then we define the groupoid C^*-algebra to be $C^*(G) = C_0(G^{(0)}) \rtimes G$. Furthermore, $C^*(S)^\wedge$ is exactly the dual $\widehat{S}$ of S as described in Section 2.2. Under these conditions it follows immediately from Theorem 6.49 that the induction map is open and factors to the desired homeomorphism. $\square$

We can actually use the machinery developed in Chapter 6 to do better though. We would like to remove the assumption that $G^{(0)}/G$ is T_0. However, we have the following proposition to consider.

Proposition 7.2. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system. Furthermore, suppose the stabilizer subgroupoid S is abelian and varies continuously. Then the following are equivalent:*

(a) $G^{(0)}/G$ *is* T_0.

(b) $C^*(G)$ *is GCR.*

(c) $C^*(G)$ *is Type I.*

Proof. Since abelian groups are GCR, the fact that (a) is equivalent to (b) follows from [Cla07, Theorem 1.1]. The fact that (b) and (c) are equivalent for separable C^*-algebras is a well known result [Dix77, Theorem 9.1]. □

This shows that if we are to deal with the "non-T_0" case then we are going to have to work with non-Type I algebras. This means working with primitive ideals instead of the spectrum. Before we begin, let us consider the action of G on $\widehat{S}$.

Corollary 7.3. *Suppose G is a second countable, locally compact Hausdorff groupoid with a Haar system and that the stabilizer subgroupoid S is abelian and continuously varying. Then then there is a strongly continuous action of G on $\widehat{S}$ given for $\gamma \in G$ and $\omega \in \widehat{S}$ by*

$$\gamma \cdot \omega(s) = \omega(\gamma^{-1} s \gamma). \tag{7.1}$$

Proof. Of course, the existence of such an action is shown in Corollary 6.32. Furthermore the action is strongly continuous in this case because the structure map for $\widehat{S}$, namely $\hat{p}$, is open since $\widehat{S}$ is continuously varying. Finally we use Proposition 6.33 to see that the action is given by (7.1). □

Remark 7.4. Lemma 6.28 still holds even if $G^{(0)}/G$ is not T_0 and in particular we will continue to make the identifications of Remark 6.29. We will regularly confuse $\operatorname{Ind}_S^G \omega$ and $\operatorname{Ind}_{S_u}^G \omega$.

Now, if $G^{(0)}/G$ is not T_0 then we cannot use Proposition 6.24 to conclude that $\operatorname{Ind}_S^G \omega$ is irreducible if $\omega \in \widehat{S}$. It is the main result in [IW09] that every representation of a second countable groupoid induced from an irreducible representation of a stability group is irreducible, even when the stabilizers are non-abelian. However, in the abelian case this result is much closer to the surface and we will give an account here.

Proposition 7.5 ([PSMW96, Lemma 2.5])**.** *Let G be a second countable locally compact Hausdorff groupoid with a Haar system and suppose the stabilizer subgroupoid S is abelian. Then $\operatorname{Ind}_{S_u}^G \omega$ is irreducible for all $\omega \in \widehat{S}_u$.*

Proof. By Proposition 6.39, it suffices to show that the representation N^ω is irreducible. First, since ω acts on $\mathbb{C}$, N^ω acts on $L^2(G_u/S_u, \sigma^u)$. Furthermore, given $f \in C_c(S_u)$ we can compute that

$$\begin{aligned} N^\omega(f)\phi([\gamma]) &= \int_G \omega(\delta(\gamma))\,\mathrm{id}_\gamma^{-1}(f(\eta))\overline{\omega(\delta(\eta^{-1}\gamma))}\phi([\eta^{-1}\gamma])d\lambda^{r(\gamma)}(\eta) \\ &= \int_G \overline{\omega(\delta(\eta^{-1}\gamma)\delta(\gamma)^{-1})}f(\eta)\phi([\eta^{-1}\gamma])d\lambda^{r(\gamma)}(\eta) \end{aligned}$$

where δ is as in Remark 6.38. Furthermore, as in Remark 6.40, we use the range map to identify G_u/S_u and $G \cdot u$ as Borel spaces, push the measure σ^u forward to a measure σ^u_* on $G \cdot u$, and view N^ω as acting on $L^2(G \cdot u, \sigma^u_*)$ via

$$N^\omega(f)\phi(\gamma \cdot u) = \int_G \overline{\omega(\delta(\eta^{-1}\gamma)\delta(\gamma)^{-1})}f(\eta)\phi(\eta^{-1}\gamma \cdot u)d\lambda^{r(\gamma)}(\eta).$$

It is straightforward to show that $\overline{\omega(\delta(\eta^{-1}\gamma)\delta(\gamma)^{-1})}$ only depends on $v = \gamma \cdot u$ and η. We write $\theta(\eta, v)$ for the corresponding Borel function. In particular, this allows us to rewrite N^ω as

$$N^\omega(f)\phi(v) = \int_G \theta(\eta, v)f(\eta)\phi(\eta^{-1} \cdot v)d\lambda^v(\eta). \tag{7.2}$$

Let N^u be the representation of $C_0(G^{(0)})$ on $L^2(G \cdot u, \sigma^u_*)$ defined in Lemma 6.41 and recall that $N^\omega(f \cdot g) = N^u(f)N^\omega(g)$ for all $f \in C_0(G^{(0)})$ and $g \in C_c(G)$. Now suppose $T \in N^u(C_0(G^{(0)}))''$ and P is a projection commuting with $N^\omega(C^*(G))$. Then in particular

$$PN^u(f)N^\omega(g)h = PN^\omega(f \cdot g)h = N^\omega(f \cdot g)Ph = N^u(f)PN^\omega(g)h.$$

Since N^ω is nondegenerate, this implies that P is in the commutant of $N^u(C_0(G^{(0)}))$ so that P commutes with T. Since $C_c(G^{(0)})$ separates points of $G \cdot u$, the von Neumann algebra $N^u(C_0(G^{(0)}))''$ is a maximal abelian subalgebra of operators on $L^2(G \cdot u, \sigma^u_*)$. It follows that any projection commuting with $N^\omega(C^*(G))$ must be of the form $N^u(f)$ where $f = \chi_E$ is the characteristic function of some set $E \subset G \cdot u$ and where we have extended N^u to $L^\infty(G)$ in the obvious fashion. It is easy to see that $N^u(f)$ still commutes with every $N^\omega(g)$. Thus for each $g \in C_c(G)$ we have

$$\begin{aligned} N^u(f)N^\omega(g)\phi(v) &= f(v)\int_G \theta(\eta, v)g(\eta)\phi(\eta^{-1} \cdot v)d\lambda^v(\eta) \\ &= \int_G \theta(\eta, v)g(\eta)f(\eta^{-1} \cdot v)\phi(\eta^{-1} \cdot v)d\lambda^v(\eta) = N^\omega(g)N^u(f)\phi(v) \end{aligned}$$

for σ_*^u-almost all v. This suffices to show that f is constant almost everywhere on $G \cdot u$ and that the projection $N^\omega(f)$ is a multiple of the identity. Therefore, the only projections commuting with $N^\omega(C^*(G))$ are the scalars and N^ω is irreducible. $\square$

Thus, even when $G^{(0)}/G$ is not T_0, we can still induce representations from $\widehat{S}$ to elements in the spectrum of $C^*(G)$.

Proposition 7.6. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system. Furthermore suppose the isotropy subgroupoid S is continuously varying and abelian. Then $\Phi : \widehat{S} \to C^*(G)^\wedge$ defined by $\Phi(\omega) = \operatorname{Ind}_S^G \omega$ is continuous and open.*

Proof. It follows from Proposition 7.5 that Φ maps into $C^*(G)^\wedge$. Furthermore, just as in Proposition 6.27, the continuity of Φ follows from the general theory of Rieffel induction. All that is left to do is show Φ is open. This proof is almost exactly the same as the openness calculation in the proof of Theorem 6.49. Suppose $\operatorname{Ind} \omega_i \to \operatorname{Ind} \omega$ in $C^*(G)^\wedge$. Since Res_M is continuous, it follows that

$$I_i = \operatorname{Res}_M \ker \operatorname{Ind}_S^G \omega_i \to I = \operatorname{Res}_M \ker \operatorname{Ind}_S^G \omega.$$

Now, the spectrum of $C^*(S)$ is Hausdorff so that we can identify $\widehat{S}$ and $\operatorname{Prim} C^*(S)$. In particular, under this identification Lemma 6.46 tells us that

$$I = \bigcap_{\gamma \in G_u} \gamma \cdot \omega, \quad \text{and} \quad I_i = \bigcap_{\gamma \in G_{u_i}} \gamma \cdot \omega_i$$

for all i. Hence, the closed set associated to I is $\overline{G \cdot \omega}$ and the closed set associated to I_i is $\overline{G \cdot \omega_i}$ for all i. Since $\omega \in \overline{G \cdot \omega}$ it follows from Lemma 6.48 that, after passing to a subnet and relabeling, there exists $\chi_i \in \overline{G \cdot \omega_i}$ such that $\chi_i \to \omega$.

Let $u = \hat{p}(\omega)$, $u_i = \hat{p}(\omega_i)$ for all i, and $\mathcal{U}$ be a neighborhood basis of ω. For each $U \in \mathcal{U}$ there exists an i_0 such that $i \geq i_0$ implies that $\chi_i \in U$. We let

$$M := \{(U, i) : U \in \mathcal{U}, \chi_i \in U\}.$$

and direct M by decreasing U and increasing i. Then M is a subnet of i such that $\chi_i \in U$ for all $(U, i) \in M$. Given $(U, i) \in M$ since U is an open set containing χ_i there exists $\gamma_{(U,i)} \in G_{u_i}$ such that $\gamma_{(U,i)} \cdot \omega_i \in U$. Now, given any $U_0 \in \mathcal{U}$ choose i_0 so that $\chi_{i_0} \in U$ and $(U_0, i_0) \in M$. If $(U, i) \in M$ such that $(U_0, i_0) \leq (U, i)$ then $\gamma_{(U,i)} \cdot \omega_i \in U \subset U_0$. Thus

$$\gamma_{(U,i)} \cdot \omega_i \to \omega.$$

This suffices to show that Φ is open. $\square$

Of course, as we said before, we need to be working with primitive ideals so we will make the switch now.

Corollary 7.7. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system. Furthermore suppose the isotropy subgroupoid S is continuously varying and abelian. Then $\Psi : \widehat{S} \to \operatorname{Prim} C^*(G)$ defined by $\Psi(\omega) = \ker \operatorname{Ind}_S^G \omega$ is continuous and open.*

Proof. Of course, Ψ is just Φ composed with the map $\pi \mapsto \ker \pi$. Since both maps are continuous and open, their composition must be also. □

We would like to factor Ψ to a homeomorphism and to do that we will need to get a handle on the equivalence relation determined by Ψ.

Lemma 7.8. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system. Furthermore suppose the isotropy subgroupoid S is continuously varying and abelian. Then $\Psi(\omega) = \Psi(\chi)$ if and only if $\overline{G \cdot \omega} = \overline{G \cdot \chi}$.*

Proof. If $\ker \operatorname{Ind}_S^G \chi = \ker \operatorname{Ind}_S^G \omega$ then we must have

$$\operatorname{Res}_M \ker \operatorname{Ind}_S^G \chi = \operatorname{Res}_M \ker \operatorname{Ind}_S^G \omega.$$

However, it now follows from Lemma 6.46, after identifying $\widehat{S}$ and $\operatorname{Prim} C^*(S)$, that

$$\bigcap_{\gamma \in G_{\hat{p}(\omega)}} \gamma \cdot \omega = \bigcap_{\gamma \in G_{\hat{p}(\chi)}} \gamma \cdot \chi.$$

This implies that the closed sets in $\widehat{S}$ associated to these ideals must be the same. Hence $\overline{G \cdot \omega} = \overline{G \cdot \chi}$.

For the reverse direction suppose that $\overline{G \cdot \omega} = \overline{G \cdot \chi}$. This implies that there exists γ_i such that $\chi = \lim_i \gamma_i \cdot \omega$. It follows from Proposition 6.43 that $\operatorname{Ind}_S^G \omega$ is equivalent to $\operatorname{Ind}_S^G \gamma \cdot \omega$ for all γ. Thus Ψ is G-invariant and, since Ψ is continuous, we get $\Psi(\gamma_i \cdot \omega) = \Psi(\omega) \to \Psi(\chi)$. Thus $\Psi(\chi) \in \overline{\{\Psi(\omega)\}}$ and, by definition of the hull-kernel topology, $\Psi(\omega) \subset \Psi(\chi)$. Reversing the roles of ω and χ above will yield the other inclusion. □

Technically the next definition uses induction for ideals, which we haven't actually introduced. All the reader needs to know is that it is characterized by the formula $\operatorname{Ind}_H^G \ker \pi = \ker \operatorname{Ind}_H^G \pi$.

Definition 7.9. Let G be a second countable locally compact Hausdorff groupoid with a Haar system. We say that G is *EH-regular* if every primitive ideal is induced from an isotropy subgroup. That is, given $P \in \operatorname{Prim} C^*(G)$ there exists $u \in G^{(0)}$ and $Q \in \operatorname{Prim} C^*(S_u)$ such that $P = \operatorname{Ind}_{S_u}^G P$.

Of course, we have already met a large class of groupoids which are EH-regular.

Corollary 7.10. *If G is a second countable locally compact Hausdorff groupoid and $G^{(0)}/G$ is T_0 then G is EH-regular.*

Proof. Theorem 6.22 tells us that every irreducible representation is induced from a stabilizer, which of course implies that every primitive ideal is induced from a stabilizer. □

Of course, the whole point is to get away from the T_0 case so we cite the following result.

Theorem 7.11 ([IW08, Theorem 2.1])**.** *Assume that G is a second countable, locally compact Hausdorff groupoid with a Haar system. If G is amenable then every primitive ideal is induced from a stability group. In other words, G is EH-regular.*

Remark 7.12. If this theorem leaves something to be desired it is that, as we saw in Section 1.3, groupoid amenability is not a transparent condition. It is worth noting that not all principal groupoids are amenable so that in particular not all groupoids with abelian, continuously varying stabilizer are amenable. Thus, the amenability condition in Theorem 7.15 below is not superfluous.

This theorem allows us to give a strengthening of Theorem 6.49 in the scalar case, however we need to, briefly, introduce a new construction.

Definition 7.13. If X is a topological space, then the T_0-*ization* of X is the quotient space $(X)^{T_0} := X/\sim$ where $\sim$ is the equivalence relation on X defined by $x \sim y$ if $\overline{\{x\}} = \overline{\{y\}}$. We equip $(X)^{T_0}$ with the quotient topology.

We will not use the following lemma and do not provide a proof, but it sheds some light on the T_0-ization definition.

Lemma 7.14 ([Wil07, Lemma 6.10])**.** *If X is a topological space then $(X)^{T_0}$ is a T_0 space. If Y is any T_0 topological space and if $f : X \to Y$ is continuous then there is a continuous map $f' : (X)^{T_0} \to Y$ such that $f = f' \circ q$ where $q : X \to (X)^{T_0}$ is the quotient map.*

Now we have enough technology to prove the main result of this section.

Theorem 7.15. *Suppose G is a second countable locally compact Hausdorff groupoid with a Haar system and that the stabilizer subgroupoid S is abelian and continuously varying. If G is EH-regular, and in particular if G is amenable or $G^{(0)}/G$ is T_0, then the map $\Psi : \widehat{S} \to \operatorname{Prim} C^*(G)$ such that $\Psi(\omega) = \ker \operatorname{Ind}_S^G \omega$ factors to a homeomorphism of $\operatorname{Prim} C^*(G)$ with $(\widehat{S}/G)^{T_0}$.*

Proof. It follows from Corollary 7.7 that Ψ is continuous and open. Furthermore, once we identify $\widehat{S}$ with $\operatorname{Prim} C^*(S)$, it is clear that Ψ is surjective if G is EH-regular, which, by Theorem 7.11, occurs whenever G is amenable, or, by Corollary 7.10, when $G^{(0)}/G$ is T_0. Finally, it is straightforward to show that $\overline{G \cdot \omega} = \overline{G \cdot \chi}$ in $\widehat{S}$ if and only if $\overline{\{G \cdot \omega\}} = \overline{\{G \cdot \chi\}}$ in $\widehat{S}/G$. Thus it follows from Lemma 7.8 that the factorization of Ψ to $(\widehat{S}/G)^{T_0}$ is injective and therefore a homeomorphism. □

Remark 7.16. If $G^{(0)}/G$ is T_0 then G is EH-regular by Corollary 7.10, and $C^*(G)$ is Type I by Proposition 7.2. In particular $\pi \mapsto \ker \pi$ is a homeomorphism of $C^*(S)^\wedge$ onto $\operatorname{Prim} C^*(S)$ so that in this case Theorem 7.15 reduces to Corollary 7.1.

As in Section 6.3 we get the following corollary.

Corollary 7.17. *If G is a second countable, locally compact, EH-regular, principal groupoid with a Haar system then* $\operatorname{Prim} C^*(G)$ *is homeomorphic to* $(G^{(0)}/G)^{T_0}$.

Proof. Since G is principal it has trivial, and hence continuously varying, abelian stabilizer $S = G^{(0)}$ and the result follows from Theorem 7.15. □

Remark 7.18. While requiring the stabilizer subgroupoid S to have a Haar system is natural from a groupoid point of view, it is a strong assumption. In particular, as will see in Section 7.2, it is this assumption which prevents Theorem 7.15 from completely generalizing [Wil07, Theorem 8.39]. However, removing this hypothesis is a serious challenge in that, unless S is continuously varying, $C^*(S)$ and $\widehat{S}$ do not exist as we have defined them.

7.2 Transformation Groupoids Redux

The purpose of this section is threefold. First, we would like to apply Theorem 7.15 to groupoid actions and restate the results in terms of the transformation groupoid algebra. Second, we would like to show that Theorem 7.15 is a partial generalization of the results in [Wil07, Section 8.3]. Finally, we will present two examples which show how we can use this theory. The first task is straightforward. Recall from Proposition 1.81 that the stabilizers and orbit space of a groupoid action appear naturally as the stabilizers and orbit space of the transformation groupoid. Proving statements about transformation groupoids often requires little more than rewording corresponding statements for groupoids. For instance EH-regularity is remolded into the following

Definition 7.19. Let G be a second countable, locally compact Hausdorff groupoid with a Haar system acting continuously on a second countable, locally compact Hausdorff space X. We say that (G, X) is *EH-regular* if every primitive ideal is induced

from an isotropy subgroup. That is, given $P \in \operatorname{Prim} C^*(G,X)$ then there exists $x \in X$ and $Q \in \operatorname{Prim} C^*(G_x)$ such that $P = \operatorname{Ind}_{G_x}^G Q$.

Observe that the stabilizer G_x is just the stabilizer of the transformation groupoid $G \ltimes X$ at x and Definition 7.19 is exactly the same as requiring that $G \ltimes X$ be EH-regular. With this in mind the following corollary is unsurprising.

Corollary 7.20. *Suppose G is a second countable, locally compact Hausdorff groupoid with a Haar system acting continuously on a second countable, locally compact Hausdorff space X. Furthermore, suppose that the stabilizer group bundle of the action S is abelian and varies continuously. Then there is an action of G on $\widehat{S}$ defined for $\gamma \in G$ and $\omega \in \widehat{S}$ by*

$$\gamma \cdot \omega(s) = \omega(\gamma^{-1} s \gamma). \tag{7.3}$$

Furthermore, if (G,X) is EH-regular, which holds if (G,X) is amenable or X/G is T_0, then the map $\Psi : \widehat{S} \to \operatorname{Prim} C^*(G,X)$ such that $\Psi(\omega) = \ker \operatorname{Ind}_S^{G\ltimes X} \omega$ factors to a homeomorphism of $\operatorname{Prim} C^*(G,X)$ with $(\widehat{S}/G)^{T_0}$.

Proof. Since G and X are second countable, the transformation groupoid is as well. Since G has a Haar system the transformation groupoid does. Furthermore the stabilizer group bundle of the action is defined to be the stabilizer subgroupoid of $G \ltimes X$ and is abelian and continuously varying by assumption. As we noted above, the condition that (G,X) is EH-regular is equivalent to requiring that $G \ltimes X$ is EH-regular. In Definition 1.101 we say that (G,X) is amenable if and only if $G \ltimes X$ is and since G and $G \ltimes X$ have the same action on X it is clear that $X/G = X/(G \ltimes X)$. At this point we have everything we need to apply Theorem 7.15. The only thing that is not clear is what we mean by $\widehat{S}/G$. Observe that by composing $\hat{p}$ with the range map $r : X \to G^{(0)}$ we get a range map on $\widehat{S}$. Suppose $\omega \in S$, $\gamma \in G$ and $s(\gamma) = r(\hat{p}(\omega))$. Let $x = \hat{p}(\omega)$. Then $s(\gamma) = r(x)$ so that $(\gamma, \gamma \cdot x) \in G \ltimes X$. Furthermore, $s(\gamma, \gamma \cdot x) = x = \hat{p}(\omega)$ so that we can let $(\gamma, \gamma \cdot x)$ act on ω. We may as well *define*

$$\gamma \cdot \omega := (\gamma, \gamma \cdot \hat{p}(\omega)) \cdot \omega.$$

Then, given $s \in S_x$ we have

$$(\gamma, \gamma \cdot x) \cdot \omega(s) = \omega((\gamma, \gamma \cdot x)^{-1}(s,x)(\gamma, \gamma \cdot x)) = \omega(\gamma^{-1} s \gamma)$$

where we are being a little sloppy about distinguishing between s and (s,x). Thus the action of G on ω is given by (7.3). It is now straightforward to show that with this action $\widehat{S}$ is a continuous G-space. Furthermore if $\omega = \gamma \cdot \chi$ then $\omega = (\gamma, \gamma \cdot \hat{p}(\chi)) \cdot \chi$ by definition. Conversely, if $\omega = (\gamma, x) \cdot \chi$ then, observing that $\gamma^{-1} \cdot x = \hat{p}(\chi)$, we have $\omega = \gamma \cdot \chi$. Thus G and $G \ltimes X$ have the same orbits in S so that $S/G = S/G \ltimes X$. The corollary now follows from Theorem 7.15. $\square$

With this we have a convenient restatement of Theorem 7.15 which doesn't (directly) use the transformation groupoid. We will use this result to explore some examples later on in this section. First, though, we would like to show that Corollary 7.20 is a partial generalization of the known results for group actions. First, the relevant theorems from [Wil07] have been reproduced below.

Theorem 7.21 (Gootman-Rosenberg-Sauvageot [Wil07, Theorem 8.21])**.** *Suppose that (A, G, α) is a separable* group *dynamical system with G amenable. Then (A, G, α) is EH-regular.*

Remark 7.22. In particular, once one sorts out what EH-regularity means for group dynamical systems, this implies that any second countable, abelian transformation group (G, X) is EH-regular.

Theorem 7.23 ([Wil07, Theorem 8.39])**.** *Let (G, X) be a locally compact transformation group with G abelian. Then $\Phi : X \times \widehat{G} \to \operatorname{Prim} C^*(G)$ such that $\Phi(x, \omega) = \operatorname{Ind}_{G_x}^G(\omega|_{G_x})$, where G_x is the stabilizer at x, is continuous and open. Furthermore Φ factors through $X \times \widehat{G}/\sim$ where $(x, \omega) \sim (x, \chi)$ if $\overline{G \cdot x} = \overline{G \cdot y}$ and $\chi\overline{\omega} \in G_x^{\perp}$, and defines a homeomorphism onto its range. If (G, X) is EH-regular, which is automatic if (G, X) is second countable by the GRS-theorem, then Φ defines a homeomorphism of $X \times \widehat{G}/\sim$ onto $\operatorname{Prim} C^*(G, X)$.*

Remark. Those readers who are careful about their references will notice some minor discrepancies. The main difference is that the result in [Wil07] is stated in terms of the crossed product $C_0(X) \rtimes_{\mathrm{lt}} G$. Of course, we saw in Section 4.4 that this is isomorphic to $C^*(G, X)$ and we will not distinguish between the two here. □

Before we begin our analysis in earnest let us make two remarks.

Remark 7.24. First, let us consider the problem of separability. As we noted in Remark 3.120, groupoid crossed products and groupoid algebras are heavily dependent on separability hypothesis. We would not expect to be able to reproduce Theorem 7.23 in the nonseparable case using groupoids. Second, let us consider amenability. As we noted in Remark 7.12, the amenability hypothesis in Theorem 7.15, and hence Corollary 7.20, are not specious. On the other hand, while Theorem 7.21 does have an amenability hypothesis this assumption will disappear in Theorem 7.23 because abelian groups are always amenable.

Remark 7.25. We should also mention the most important difference between Theorem 7.23 and Corollary 7.20. Suppose a group G acts on a space X. If G is abelian then all of the stabilizers are abelian. However, they certainly don't have to vary continuously. In particular Theorem 7.23 holds for non-continuously varying stabilizers. On the other hand, even in the transformation group case, Corollary 7.20 doesn't

make sense if S doesn't vary continuously. This is an unsatisfactory aspect of the current theory and it is an open question if/how it can be addressed. Let us finish by pointing out that Corollary 7.20 does have its uses. In Example 7.28 we present an action of a *nonabelian* group with continuously varying abelian stabilizers. This action can be studied using Corollary 7.20 but is outside the scope of Theorem 7.23.

So, we cannot fully reproduce Theorem 7.23 with our current theory. Our goal will be to show that given a transformation group G such that G is abelian *and* has continuously varying stabilizers then Theorem 7.23 and Corollary 7.20 say the same thing.

Proposition 7.26. *Let G be a second countable, locally compact Hausdorff abelian group acting on a second countable, locally compact Hausdorff space X. Furthermore suppose that the stabilizers vary continuously in G. Then $X \times \widehat{G}/\sim$ is naturally homeomorphic to $(\widehat{S}/G)^{T_0}$ so that Theorem 7.23 and Corollary 7.20 have the same conclusion.*

Proof. First, observe that if G is abelian then all of the stabilizers are abelian. Thus (G, X) satisfies both the requirements of Theorem 7.23 and Corollary 7.20. Furthermore, the GRS-theorem implies that (G, X) is EH-regular so that this condition is satisfied for both Theorem 7.23 and Corollary 7.20. All that is left is to show that $X \times \widehat{G}/\sim$ and $(\widehat{S}/G)^{T_0}$ are naturally isomorphic so that in this case each theorem can be obtained from the other.

Define $\rho : X \times \widehat{G} \to \widehat{S}$ by $\rho(x,\omega) = \omega|_{S_x}$. It is straightforward to use Proposition 2.38 to show that ρ is continuous. It is a classical result [Rud62] that the dual of a subgroup is isomorphic to a quotient of the dual of the full group via restriction. Hence characters on subgroups can be extended to characters on the full group and ρ is a surjection. Define $\equiv$ on $X \times \widehat{G}$ by $(x,\omega) \equiv (y,\chi)$ if and only if $x = y$ and $\chi\overline{\omega} \in G_x^{\perp}$. It is clear from [Rud62] that $\rho(x,\omega) = \rho(y,\chi)$ if and only if $(x,\omega) \equiv (y,\chi)$. Thus, if we can show ρ is open, it will follow that it factors to a homeomorphism from $X \times \widehat{G}/\equiv$ onto $\widehat{S}$. Suppose $\rho(x_i,\omega_i) \to \rho(x,\omega)$. Since $\hat{p}(\rho(y,\chi)) = y$ for all $(y,\chi) \in X \times \widehat{G}$ we clearly have $x_i \to x$. Recall from [Wil07, Lemma 2.35] that, after identifying $C^*(G,X)$ with $C_0(X) \rtimes G$, there is a natural map $\iota : C^*(G) \to M(C^*(G,X))$. This map induces a restriction map Res_G from representations of $C^*(G,X)$ to $C^*(G)$. In particular, as with all such restriction maps, it is a continuous process and we have

$$\mathrm{Res}_G \mathrm{Ind}_S^{G\ltimes X} \omega_i|_{G_{x_i}} \to \mathrm{Res}_G \mathrm{Ind}_S^{G\ltimes X} \omega|_{G_x}.$$

Basically, what is going on is that $\mathrm{Ind}_S^{G\ltimes X} \chi|_{G_y}$ is a representation of $C^*(G,X) \cong C_0(X) \rtimes G$ for all $(y,\chi) \in X \times \widehat{G}$, and as such must be the integrated form of some covariant representation (π, U). The restriction map Res_G gives us the unitary

part U. However, [Wil07, Corollary 5.6] says that this unitary part is equivalent to $\operatorname{Ind}_{G_y}^G \chi|_{G_y}$. Thus, putting it all together, plus a little more, we have

$$\ker \operatorname{Ind}_{G_{x_i}}^G \omega_i|_{G_{x_i}} \to \ker \operatorname{Ind}_{G_x}^G \omega|_{G_x}. \tag{7.4}$$

It follows from [Wil07, Proposition 5.14] that given $(y,\omega) \in X \times \widehat{G}$, the closed set in $\widehat{G}$ associated to $\ker \operatorname{Ind}_{G_y}^G \chi|_{G_y}$ is $\chi G_y^\perp$. At this point we can use (7.4) and Lemma 6.48 to pass to a subnet, relabel, and find $\sigma_i \in \omega_i G_{x_i}^\perp$ such that $\sigma_i \to \omega$. It follows that $(\sigma_i, x_i) \to (\omega, x)$. Since we clearly have $(\sigma_i, x_i) \equiv (\omega_i, x_i)$ for all i this suffices to show that ϕ is open.

Using ϕ we can transport the action of G on $\widehat{S}$ to $X \times \widehat{G}/\equiv$. It is easy to show, using the fact that G is abelian, that this action is given by

$$s \cdot [x,\omega] = [s \cdot x, \omega]. \tag{7.5}$$

We would like to show that $((X \times \widehat{G}/\equiv)/G)^{T_0} = X \times \widehat{G}/\sim$. In other words, we want to see that the equivalence relation induced by the iterated quotient is exactly $\sim$. Observe that, almost by definition, (x,ω) and (y,χ) will be identified in $((X \times \widehat{G}/\equiv)/G)^{T_0}$ if and only if $\overline{G \cdot [x,\omega]} = \overline{G \cdot [y,\chi]}$. Suppose $(x,\omega) \sim (y,\chi)$. Since $\overline{G \cdot y} = \overline{G \cdot x}$ we must have s_i such that $s_i \cdot x \to y$. It follows from (7.5) that $[y,\omega] \in \overline{G \cdot [x,\omega]}$. However, $[y,\omega] = [y,\chi]$ so that we must have $\overline{G \cdot [x,\omega]} = \overline{G \cdot [y,\chi]}$. Next, consider the opposite direction. If $\overline{G \cdot [x,\omega]} = \overline{G \cdot [y,\chi]}$ then there exists a sequence $s_i \in G$ such that

$$[s_i \cdot x, \omega] \to [y,\chi].$$

It is straightforward to show, using the fact that ρ is open, that the quotient map $X \times \widehat{G} \to X \times \widehat{G}/\equiv$ is open. Use this fact to pass to a subnet, relabel, and find $\sigma_i \in \widehat{G}$ such that $(s_i \cdot x, \omega) \equiv (s_i \cdot x, \sigma_i)$ for all i and

$$(s_i \cdot x, \sigma_i) \to (y,\chi).$$

First, observe that this implies that $\overline{G \cdot x} = \overline{G \cdot y}$. Furthermore, we have $\sigma_i \overline{\omega} \in G_x^\perp$ for all i. Since this set is closed and $\sigma_i \to \chi$ we must also have $\chi\overline{\omega} \in G_x^\perp$. Thus $(x,\omega) \sim (y,\chi)$. Hence, the equivalence relation induced by the iterated crossed product is exactly $\sim$ and we have

$$(\widehat{S}/G)^{T_0} \cong ((X \times \widehat{G}/\equiv)/G)^{T_0} = X \times \widehat{G}/\sim. \qquad \square$$

7.2.1 Examples

The first example we present is of a non-abelian group action with abelian, continuously varying stabilizers. Unfortunately, it is also a transitive action. As such it really falls under the purview of [MRW87, Section 3]. However, it is elegant enough that we will include it here for reference.

Example 7.27. Let $G = \mathbb{R} \setminus \{0\} \times \mathbb{R}$ be the $ax + b$ group and let G act on $X = \mathbb{R}$ by evaluation. That is to say, we have $(ax + b) \cdot r = ar + b$. This is a classic continuous group action. Now, the $ax+b$ group is non-abelian, but it turns out that the stabilizers are abelian. Given $r \in X$ we have

$$S_r = \{ax + b \in G : ar + b = r\} = \{ax + r(1 - a) : a \in \mathbb{R} \setminus \{0\}\},$$

and it is straightforward to show that S_r is isomorphic to the multiplicative group of $\mathbb{R}$. Suppose $r_i \to r$ in X and $a \in \mathbb{R} \setminus \{0\}$. Then $ax + r_i(1 - a) \to ax + r(1 - a)$ in G and this suffices to show that the stabilizers vary continuously. Next, the orbit space X/G is a single point since G acts transitively and is trivially T_0. Thus we can use Corollary 7.20 to identify $\operatorname{Prim} C^*(G, X)$ with $(\widehat{S}/G)^{T_0}$. Fix $r \in X$ and consider the continuous map $\phi : \widehat{S}_r \to \widehat{S}/G$ defined by $\phi(\omega) = G \cdot \omega$. If $\omega, \chi \in \widehat{S}_r$ and $(ax + b) \cdot \omega = \chi$ then we must have $(ax + b) \in S_r$. However, S_r is abelian and therefore $\chi = (ax + b) \cdot \omega = \omega$. Thus ϕ is injective. Next suppose $\omega \in \widehat{S}_l$ for some $l \in X$. Then there exists $(ax + b)$ such that $(ax + b) \cdot l = r$. Thus $(ax + b) \cdot \omega \in \widehat{S}_r$ and $\phi((ax + b) \cdot \omega) = G \cdot ((ax + b) \cdot \omega) = G \cdot \omega$. Hence ϕ is surjective as well. Now suppose $G \cdot \omega_i \to G \cdot \omega$. We can pass to a subnet, relabel, and choose new representative so that $\omega_i \to \omega$. Since the transformation groupoid $G \times X$ is second countable and transitive we can cite [MRW87, Theorem 2.2A,2.2B] to conclude that the restriction of the source map on $G \times X$ to $(G \times X)^r$ is open. Since $\hat{p}(\omega_i) \to \hat{p}(\omega)$ we can pass to another subsequence, relabel, and find $(a_i x + b_i, r) \in G \times X$ and $(ax + b, r) \in G \times X$ such that $(a_i x + b_i, r) \to (ax + b, r)$, $(a_i x + b_i)^{-1} \cdot r = \hat{p}(\omega_i)$ for all i and $(ax + b)^{-1} \cdot r = \hat{p}(\omega)$. Thus we have

$$(a_i x + b_i) \cdot \omega_i \to (ax + b, r) \cdot \omega.$$

But each $(a_i x + b_i) \cdot \omega_i$ and $(ax + b) \cdot \omega$ is in S^r and they map to $G \cdot \omega_i$ and $G \cdot \omega$, respectively. This suffices to show that ϕ^{-1} is a continuous map. The upshot is that $\widehat{S}/G$ is homeomorphic to $\widehat{S}_r$. Since $\widehat{S}_r$ is already T_0, taking the T_0-ization doesn't do anything and we have the following chain of identifications

$$\operatorname{Prim} C^*(G, X) \cong (\widehat{S}/G)^{T_0} \cong \widehat{S}_r \cong \widehat{\mathbb{R}^\times} \cong \mathbb{R}^\times$$

where $\mathbb{R}^\times$ is the multiplicative group of $\mathbb{R}$ and we have used the fact that this group

is self dual.

As we noted, this example is unsatisfactory since G acts transitively. What's more, we actually used an important result from [MRW87] at one point so we would have been better off using [MRW87, Theorem 3.1] from the start. This next example is much better in that it requires the full power of Corollary 7.20.

Example 7.28. Let $G = SO(3,\mathbb{R})$ and $X = \mathbb{R}^3 \setminus \{(0,0,0)\}$. Let G act on X by rotation. Once again this is a classic continuous group action. It is clear that G is not abelian. However, it does have abelian isotropy. Given a vector $v \in X$ it's easy to see that S_v is the set of rotations about the line described by v. In particular, if we let v be the first vector of an orthogonal basis for $\mathbb{R}^3$ then it is straightforward to show that S_v is the set of matrices in $SO(3)$ which fix the first coordinate. This is isomorphic to $SO(2)$ which is itself isomorphic to the circle group and is therefore abelian. It is a little bit more complicated to see that the stabilizers vary continuously. Suppose $v_i \to v \in X$ and that S is a rotation about v. If S is rotation by θ then the goal will be to show that the rotations about v_i by θ, say S_i, converge to S. This is intuitively clear. Now, it takes some computation, but one can show that the matrix S_w^θ which rotates θ degrees around a vector $w = (x,y,z)$ is given by

$$\frac{1}{L^2}\begin{bmatrix} x^2+(y^2+z^2)\cos\theta & xy(1-\cos\theta)-zL\sin\theta & xz(1-\cos\theta)+yL\sin\theta \\ xy(1-\cos\theta)+zL\sin\theta & y^2+(x^2+z^2)\cos\theta & yz(1-\cos\theta)-xL\sin\theta \\ xz(1-\cos\theta)-yL\sin\theta & yz(1-\cos\theta)+xL\sin\theta & z^2+(x^2+y^2)\cos\theta \end{bmatrix}$$

where $L = \sqrt{x^2+y^2+z^2}$. Observe that S_w^θ varies continuously with respect to w and hence $S_i \to S$. This shows that the stabilizer bundle is continuously varying. Next, consider X/G. With a little thought one can convince oneself that this space is homeomorphic to the open half-line. Thus X/G is obviously T_0. We may now use Corollary 7.20 to identify $\operatorname{Prim} C^*(G,X)$ with $(\widehat{S}/G)^{T_0}$. Without, going into the details we will cap this example by examining $(\widehat{S}/G)^{T_0}$. We already observed that S_v is isomorphic to $\mathbb{T}$. It is not particularly difficult, considering the matrix formula above, to show that S is isomorphic to $X \times \mathbb{T}$. Thus the dual bundle $\widehat{S}$ is isomorphic to $X \times \mathbb{Z}$. Next, let us consider the action of G on $X \times \mathbb{T}$. Suppose $S \in S_v$ is a rotation by θ around v. Given $U \in G$ a computation shows that USU^* is just rotation about Uv by θ. Thus, the action of G on $X \times \mathbb{T}$ is given by $U \cdot (v,\theta) = (Uv,\theta)$. Using this fact, it is straightforward to show that the action of G on $X \times \mathbb{Z}$ is given by $U \cdot (v,z) = (Uv,z)$. In particular, the quotient $X \times Z/G$ is isomorphic to $(0,\infty) \times \mathbb{Z}$. Since this space is clearly T_0, taking the T_0-ization doesn't do anything and we have

$$\operatorname{Prim} C^*(G,X) \cong (0,\infty) \times \mathbb{Z}.$$

Remark 7.29. It would be nice to apply the results of this section to genuine groupoid

actions. Unfortunately the field is laking in naturally defined groupoids with interesting continuously varying isotropy.

7.3 Groupoid Algebras with Hausdorff Spectrum

We finish this chapter, and the thesis, with an in depth examination of which conditions imply that groupoid C^*-algebras have Hausdorff spectrum. At the risk of spoiling the punchline, it turns out that we don't find any good conditions. In particular, we are trying to generalize [Wil82], which states that, for abelian transformation groups with T_0-orbit space, the spectrum of $C^*(G, X)$ is Hausdorff if and only if the stabilizers vary continuously and the orbit space is Hausdorff. It turns out that the naive generalization of [Wil82] doesn't work and the situation is more complex. Of course, this is in many ways more interesting than if the straightforward generalization would have held. There is a notion that "everything which is true for transformation groups is true for groupoids" and this provides a situation where such wishful thinking fails.

For now, let us drop down a couple of "levels" and consider the problem of when the spectrum of $C^*(G)$ is T_0. Suppose we are working with a groupoid that has continuously varying abelian stabilizers. If $G^{(0)}/G$ is T_0 then it follows from Proposition 7.2 that $C^*(G)$ is Type I. Hence its spectrum is isomorphic to its primitive ideal space and must be T_0. Furthermore, if $G^{(0)}/G$ is T_0 it follows from Corollary 7.1 that $C^*(G)^\wedge$ is homeomorphic to $\widehat{S}/G$. Thus we can make the interesting deduction that $\widehat{S}/G$ is T_0 whenever $G^{(0)}/G$ is T_0. There is actually a direct proof for this which can be generalized to the T_1 case.

Proposition 7.30. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system and continuously varying abelian stabilizers S. If $G^{(0)}/G$ is T_0 (resp. T_1) then S/G and $\widehat{S}/G$ are T_0 (resp. T_1) as well.*

Proof. First, recall that G acts on S by conjugation so that $\gamma \cdot s = \gamma s \gamma^{-1}$. Suppose we are given $[s], [t] \in S/G$ such that $[s] \neq [t]$. Let $\tilde{p}$ denote the factorization of p to S/G and set $[u] = \tilde{p}([s])$ and $[v] = \tilde{p}([t])$. Suppose $[u] \neq [v]$. If $G^{(0)}/G$ is T_0 then we can find an open set O in $G^{(0)}/G$ containing either $[u]$ or $[v]$ and not the other. Clearly $\tilde{p}^{-1}(O)$ is an open set containing either $[s]$ or $[t]$ and not the other. If $G^{(0)}/G$ is T_1 then we can find open sets U, V such that $[u] \in U$, $[v] \in V$ and $[u] \notin V$, $[v] \notin U$. Then clearly $\tilde{p}^{-1}(U)$ contains $[s]$ and not $[t]$ and $\tilde{p}^{-1}(U)$ contains $[t]$ and not $[s]$. Thus, when $[u] \neq [v]$ we can separate $[s]$ and $[t]$ to the same degree that we can separate $[u]$ and $[v]$.

Next, suppose $[u] = [v]$. We can assume without loss of generality that $s, t \in S_u$. However, since $[s] \neq [t]$, we must have $s \neq t$. Let $q : S \to S/G$ be the quotient map

and recall that it is open. Fix a neighborhood U of s. If $t \notin G \cdot U$ then $[t] \notin q(U)$ and $q(U)$ separates $[s]$ from $[t]$. Now suppose that $t \in G \cdot U$ for all neighborhoods U of s. Then for each U there exists $\gamma_U \in G$ and $s_U \in U$ such that $s_U = \gamma_U \cdot t$. If we direct s_U by decreasing U then it is clear that $s_U \to s$. This implies that

$$\gamma_U \cdot u = r(\gamma_U) = p(s_U) \to u. \tag{7.6}$$

Since $G^{(0)}/G$ is (at least) T_0 we can use Theorem 5.4 to conclude that $[\gamma] \mapsto r(\gamma)$ is a homeomorphism from G_u/S_u to $[u]$. It follows from (7.6) that $[\gamma_U] \to [u]$ in G_u/S_u. However, the quotient map from G_u onto G_u/S_u is open so that we can pass to a subnet, relabel, and choose $r_U \in S_u$ such that $\gamma_U r_U \to u$. Hence $\gamma_U r_U \cdot t \to u \cdot t = t$. But S_u is abelian so that $r_U \cdot t = t$ for all U. Therefore we also have $\gamma_U r_U \cdot t = \gamma_U \cdot t = s_U \to s$. But then $s = t$, which is a contradiction. It follows that we must have been able to separate $[s]$ from $[t]$. This argument is completely symmetric so that we can also find an open set around $[t]$ which does not contain $[s]$ (even if $G^{(0)}/G$ is only T_0). It now follows that S/G is T_0 (resp. T_1) if $G^{(0)}/G$ is T_0 (resp. T_1). The argument for $\widehat{S}$ is exactly the same and we end up with the same result. □

This gives us the following provocative corollary.

Corollary 7.31. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system and abelian continuously varying stabilizers S. Then $C^*(G)^\wedge$ is T_0 if $G^{(0)}/G$ is T_0 and $C^*(G)^\wedge$ is T_1 if $G^{(0)}/G$ is T_1.*

Proof. As long as $G^{(0)}/G$ is at least T_0 we can use Corollary 7.1 to identify $C^*(G)^\wedge$ with $\widehat{S}/G$. The result now follows from Proposition 7.30. □

If we could extend this result to the T_2 (i.e. Hausdorff) case then we would have made great progress in identifying when the spectrum of $C^*(G)$ is Hausdorff. In particular we would have generalized one direction of the main result in [Wil82] which states that, for abelian transformation groups, the transformation group algebra has Hausdorff spectrum if and only if X/G is Hausdorff and the stabilizers vary continuously. Interestingly enough, it turns out that Proposition 7.30, and hence Corollary 7.31, doesn't extend to the Hausdorff case. In order to build our counterexample we will have to use Green's famous example from [Gre77].

Example 7.32. The space $X \subset \mathbb{R}^3$ will consist of countably many orbits, with the points $x_0 = (0,0,0)$ and $x_n = (2^{-2n}, 0, 0)$ for $n \in \mathbb{N}$ as a family of representatives. The action of $\mathbb{R}$ on X is described by defining maps $\phi_n : \mathbb{R} \to X$ such that $\phi_n(s) = s \cdot x_n$. In particular we let

$$\phi_0(s) = (0, s, 0)$$

and for $n \geq 1$

$$\phi_n(s) = \begin{cases} (2^{-2n}, s, 0) & s \leq n \\ (2^{-2n} - (s-n)2^{-2n-1}, n\cos(\pi(s-n)), n\sin(\pi(s-n))) & n < s < n+1 \\ (2^{-2n-1}, s-1-2n, 0) & s \geq n+1. \end{cases}$$

This is a well known example of a continuous, free group action that is not proper. It is straightforward to observe that the orbit space $X/\mathbb{R}$ is homeomorphic to the subset $\{x_n\}_{n=0}^{\infty}$ of $\mathbb{R}^3$. We will also make use of the restriction of this action to an action of $\mathbb{Z}$ on the subset

$$Y = \{\phi_n(m) : n \in \mathbb{N}, m \in \mathbb{Z}\}.$$

In particular, the restriction to an action of $\mathbb{Z}$ on X clearly yields a continuous action, as does a further restriction to the action of $\mathbb{Z}$ on the $\mathbb{Z}$-invariant subset Y.

Next, we will build an example of a groupoid G with continuously varying stabilizers such that $G^{(0)}/G$ is Hausdorff and both S/G and $\widehat{S}/G$ are not Hausdorff. In particular, since $G^{(0)}/G$ is definitely T_0, it will follow that $C^*(G)^\wedge$ is not Hausdorff.

Example 7.33. Let X be as in Example 7.32 and let $G = \mathbb{R} \rtimes_\phi \mathbb{R}$ be the semidirect product where we define $\phi(r)(s) := se^r$. Note that $\phi(r)$ is clearly a continuous automorphism of $\mathbb{R}$. We also have

$$\phi(r+t)(s) = se^{r+t} = se^r e^t - \phi(r)(\phi(t)(s)).$$

Thus ϕ is a homomorphism into the automorphism group and it follows that the semidirect product is a well defined, second countable, locally compact Hausdorff group. The group operations are given by

$$(r,s)(u,v) = (r + \phi(s)(u), s+v) = (r + ue^s, s+v),$$
$$(r,s)^{-1} = (\phi(-s)(-r), -s) = (-re^{-s}, -s).$$

Next, let the second factor of G act on X as in Example 7.32. In other words, let $(r,s) \cdot x := s \cdot x$ where $s \cdot x$ is defined via the ϕ_n. It is straightforward to show that this is a continuous group action. It follows that the transformation groupoid $G \ltimes X$ is a second countable, locally compact Hausdorff groupoid with a Haar system. Furthermore, the stabilizer group at x is $S_x = \{(s,0) : s \in \mathbb{R}\}$ for all $x \in X$. Since the stabilizers are constant, they must vary continuously, both in G and in $G \ltimes X$. Furthermore

$$(s,0)(t,0) = (s + e^0 t, 0) = (s+t, 0)$$

and this clearly implies that S_x is abelian. Thus $G \ltimes X$ has continuously varying abelian stabilizers. Furthermore, the action of $G \ltimes X$ on X has the same orbits as

the action of G on X which in turn has the same orbits as the action of $\mathbb{R}$ on X. In particular X/G is homeomorphic to $\{x_i\}_{i=0}^{\infty}$ which is clearly Hausdorff.

Let S be the stabilizer subgroupoid of $G \ltimes H$ and $s_n = \{((e^{-2n-1}, 0), (2^{-2n}, 0, 0))\}$. Then $\{s_n\} \subset S$ and clearly $s_n \to s = ((0,0),(0,0,0))$. Consider

$$\gamma_n = ((0, 2n+1), (2^{-2n-1}, 0, 0)) \quad \text{for all } n.$$

One can compute that

$$s(\gamma_n) = (2^{-2n}, 0, 0), \quad \text{and} \quad r(\gamma_n) = (2^{-2n-1}, 0, 0).$$

In particular if we let γ_n act on s_n then we obtain

$$\begin{aligned} \gamma_n \cdot s_n &= ((0, 2n+1)(e^{-2n-1}, 0)(0, -2n-1), (2^{-2n-1}, 0, 0)) \\ &= ((1, 0), (2^{-2n-1}, 0, 0)). \end{aligned}$$

Therefore we have $\gamma_n \cdot s_n \to t = ((1,0),(0,0,0))$. This, of course, implies that $[s_n]$ converges to both $[s]$ and $[t]$ in $S/G \ltimes X$. If $\gamma \cdot s = t$ then we would have $r(\gamma) = s(\gamma) = (0,0,0)$ so that $\gamma \in S_{(0,0,0)}$. In particular, $\gamma = ((h,0),(0,0,0))$ for some $h \in \mathbb{R}$. But if this is the case then it is easy to compute that $t = \gamma \cdot s = s$. This is a contradiction so that we must have $[s] \neq [t]$. Hence $[s_n]$ has two distinct limits in $S/G \ltimes X$.

Next, we show that $\widehat{S}/G \ltimes X$ is not Hausdorff. First, however, we have to compute the dual. Since the stabilizers are constant in G it follows that S must be a trivial group bundle. In particular, S is isomorphic to $\mathbb{R} \times X$ via the map $((s,0),x) \mapsto (s,x)$. Thus we can identify $\widehat{S}$ with $\widehat{\mathbb{R}} \times X \cong \mathbb{R} \times X$ where we recall that $\hat{s} \in \mathbb{R}$ acts as a character on $\mathbb{R}$ via $\hat{s}(t) = e^{ist}$. There is an action of $G \ltimes X$ on $\widehat{S}$ given by $\gamma \cdot \omega(s) = \omega(\gamma^{-1} \cdot s)$. We can calculate that in our example

$$\begin{aligned} ((s,t),x) \cdot (\hat{r}, (s,t)^{-1} \cdot x)(q,x) &= (\hat{r}, -t \cdot x)(((s,t),x)^{-1} \cdot (q,x)) \\ &= (\hat{r}, -t \cdot x)(qe^{-t}, -t \cdot x) \\ &= e^{irqe^{-t}} = (\widehat{re^{-t}}, x)(q,x). \end{aligned}$$

It follows that

$$((s,t),x) \cdot (\hat{r}, -t \cdot x) = (\widehat{re^{-t}}, x) \tag{7.7}$$

First we observe the following from (7.7). Suppose $((s,t),x) \cdot (\hat{r}, x) = (\hat{q}, x)$. Then we must have $(s,t) \in S_x$ and therefore $t = 0$. But then

$$(\hat{q}, x) = ((s,t),x) \cdot (\hat{r}, x) = (\widehat{re^{0}}, x).$$

In particular, the action of $G \ltimes X$ is trivial when restricted to a fixed fibre in $\widehat{S}$. Moving on, let $\gamma_n \in G \ltimes X$ be as above and $\omega_n = (\hat{1}, (2^{-2n}, 0, 0))$ for all n. It is clear that $\omega_n \to \omega = (\hat{1}, (0, 0, 0))$. However, it follows from (7.7) that

$$\gamma_n \cdot \omega_n = ((e^{-2n-1})^\wedge, (2^{-2n-1}, 0, 0)).$$

Thus $\gamma_n \cdot \omega_n \to \chi = (\hat{0}, (0, 0, 0))$ and $[\omega_n]$ converges to both $[\omega]$ and $[\chi]$ in $\widehat{S}/G \ltimes X$. Furthermore, since ω and χ are distinct elements of a single fibre, we must have $[\omega] \neq [\chi]$.

This example shows that Proposition 7.30 does not extend to the T_2 case and in particular we cannot use it to determine when the spectrum of a groupoid is Hausdorff or not. However, Example 7.33 was constructed to behave poorly, and there are large classes of groupoids for which Proposition 7.30 does extend. We would like to find an additional hypothesis which will allow us to make this extension. Consider the following

Proposition 7.34. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system and continuously varying abelian stabilizers S. Then the action of G on S factors to an action of the orbit groupoid R on S which is strongly continuous when we give R the quotient topology. Similarly the the action of G on $\widehat{S}$ factors to an action of R on $\widehat{S}$ which is strongly continuous when we give R the quotient topology.*

Proof. Let $\pi : G \to R$ be the canonical map. Define an action of the orbit groupoid R on S by factoring the action of G through π and setting

$$\pi(\gamma) \cdot s := \gamma \cdot s. \tag{7.8}$$

whenever $s(\gamma) = p(s)$. We need to show that this action is well defined. Given $\gamma, \eta \in G$ such that $\pi(\gamma) = \pi(\eta)$ and $\pi(\gamma)$ and $\pi(\eta)$ act on s, we have $\gamma^{-1}\eta \in S_{p(s)}$. In particular, since $S_{p(s)}$ is abelian, we obtain

$$\gamma \cdot s = \gamma \cdot (\gamma^{-1}\eta \cdot s) = \eta \cdot s.$$

Hence the action is well defined. It is straightforward, using the fact that π is a homomorphism, to show that the action respects the groupoid operations. Furthermore, the structure map for S is open by assumption. We would like to show that this action is continuous when we give R the quotient topology. Suppose $\pi(\gamma_i) \to \pi(\gamma)$ in R_Q and $s_i \to s$ such that $p(s_i) = s(\gamma_i)$ for all i and $p(s) = s(\gamma)$. Then, citing Proposition 1.53, we can pass to a subnet, relabel, and choose new representatives so that $\gamma_i \to \gamma$. However, this implies $\gamma_i \cdot s_i \to \gamma \cdot s$ and we are essentially done.

Next, define an action of R on $\widehat{S}$ via

$$(u, v) \cdot \omega(s) = \omega((v, u) \cdot s).$$

It is straightforward to show that this action respects the groupoid operations, and since $\widehat{S}$ varies continuously, the structure map $\hat{p}$ is open. All we need to do is show that the action is continuous when we give R the quotient topology. Suppose $\omega_i \to \omega$ and $(u_i, v_i) \to (u, v)$ in R_Q such that $\hat{p}(\omega_i) = u_i$ and $\hat{p}(\omega) = u$. Next suppose $s_i \to s$ such that $p(s_i) = v_i$ and $p(s) = v$. Then, using Proposition 2.38 and the continuity of the action of R_Q on S, we have

$$\omega_i((u_i, v_i) \cdot s_i) \to \omega((u, v) \cdot s).$$

This suffices to show that $(u_i, v_i) \cdot \omega_i \to (u, v) \cdot \omega$. □

We will see in Example 7.40 that Proposition 7.34 doesn't hold if we use the product topology instead of the quotient topology. It turns out that this fact is an obstruction to generalizing Proposition 7.30 to the Hausdorff case. Recall that we use R_P to denote the orbit groupoid equipped with the restriction of the product topology, while R_Q denotes the orbit groupoid with the quotient topology.

Proposition 7.35. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system and continuously varying abelian stabilizers. Furthermore, suppose that the action of R_P on S is continuous. Then S/G is Hausdorff if $G^{(0)}/G$ is Hausdorff. Similarly, if the action of R_P on $\widehat{S}$ is continuous and $G^{(0)}/G$ is Hausdorff then $\widehat{S}/G$ is Hausdorff.*

Proof. Suppose $[s_i] \to [s]$ and $[s_i] \to [t]$ in S/G. Let $\tilde{p}$ be the factorization of p to S/G and set $[u] = \tilde{p}([s])$ and $[v] = \tilde{p}([t])$. Suppose $[u] \neq [v]$. Then, because $G^{(0)}/G$ is Hausdorff, we can find disjoint open sets U and V which separate $[u]$ and $[v]$. Hence $\tilde{p}^{-1}(U)$ and $\tilde{p}^{-1}(V)$ are disjoint open sets which separate $[s]$ and $[t]$. However, $[s_i]$ must eventually be in both of these sets, which is a contradiction. It follows that $[u] = [v]$. We may as well assume that $s, t \in S_u$. Since the quotient map $S \to S/G$ is open, we can pass to a subsequence, twice, relabel, choose new representatives, and then find $\gamma_i \in G$ such that $s_i \to s$ and $\gamma_i \cdot s_i \to t$. Let $\pi(\gamma_i) = (u_i, v_i)$. Then $u_i = p(\gamma_i \cdot s_i) \to u$ and $v_i = p(s_i) \to u$. Hence $(u_i, v_i) \to (u, u)$ in R_P. Since the action of R_P is continuous, we have $(u_i, v_i) \cdot s_i \to (u, u) \cdot s = s$. However, the action of R is just the factorization of the action of G so that $(u_i, v_i) \cdot s_i = \gamma_i \cdot s_i \to t$. Hence $t = s$ and S/G is Hausdorff. The corresponding proof for $\widehat{S}$ is exactly the same. □

We can combine this fact with our identification of the spectrum to obtain the following corollary.

Corollary 7.36. *Let G be a second countable locally compact, Hausdorff groupoid with a Haar system and abelian continuously varying stabilizers S. If $G^{(0)}/G$ is Hausdorff and the action of R_P on $\widehat{S}$ is continuous then $C^*(G)$ has Hausdorff spectrum.*

Proof. Since $G^{(0)}/G$ is definitely T_0 we can apply Corollary 7.1. The result now follows from Proposition 7.35. $\square$

Of course, this isn't very useful unless we can prove that there are interesting groupoids for which the action of R_P on $\widehat{S}$ is continuous. We start our search by finding a number of equivalent conditions.

Proposition 7.37. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system and continuously varying abelian stabilizers S. Then the following are equivalent.*

(a) *The action of R_P on S is continuous.*

(b) *The action of R_P on $\widehat{S}$ is continuous.*

(c) *If $\{s_i\} \subset S$, $s \in S$ and $\{\gamma_i\} \subset G$ such that $s_i \to s$, $s(\gamma_i) = p(s_i)$ and $r(\gamma_i) \to p(s)$ then $\gamma_i \cdot s_i \to s$.*

(d) *If $\{\omega_i\} \subset \widehat{S}$, $\omega \in \widehat{S}$ and $\{\gamma_i\} \subset G$ such that $\omega_i \to \omega$, $s(\gamma_i) = \hat{p}(\omega_i)$ and $r(\gamma_i) \to \hat{p}(\omega)$ then $\gamma_i \cdot \omega_i \to \omega$.*

(e) *The map $S \to G^{(0)} * S/G = \{(u, [s]) : [p(s)] = [u]\}$ given by $s \mapsto (p(s), [s])$ is a homeomorphism.*

(f) *The map $\widehat{S} \to G^{(0)} * \widehat{S}/G = \{(u, [\omega]) : [\hat{p}(\omega)] = [u]\}$ given by $\omega \mapsto (\hat{p}(\omega), [\omega])$ is a homeomorphism.*

From a certain point of view, what the last two conditions in Proposition 7.37 are saying is that the topology on S and $\widehat{S}$ is somehow "constant" over G orbits.

Proof. We start by proving that (a),(c) and (e) are equivalent. First we show (a) implies (e). Given (a) let $\phi : S \to G^{(0)} * S/G$ be given by $\phi(p(s), [s])$. It is clear that ϕ is continuous. Furthermore, if $\phi(s) = \phi(t)$ then $p(s) = p(t)$ and $[s] = [t]$. But the action of G is trivial when restricted to a single fibre of S so that $s = t$. Next, if we have $(u, [s]) \in G^{(0)} * S/G$ then $[u] = [p(s)]$. In particular, there exists $\gamma \in G$ such that $r(\gamma) = u$ and $s(\gamma) = p(s)$. Then $\phi(\gamma \cdot s) = (u, [\gamma \cdot s]) = (u, [s])$. Thus ϕ is onto. It is easy to see that in general $\phi^{-1}(u, [s]) = (u, p(s)) \cdot s$. Next, suppose $(u_i, [s_i]) \to (u, [s])$. Pass to a subnet, relabel, choose new representatives, and assume $s_i \to s$. Since $(u_i, p(s_i)) \to (u, p(s))$ in R_P, we have $(u_i, p(s_i)) \cdot s_i \to (u, p(s)) \cdot s$. This suffices to show that the inverse map is continuous. Next, we show that (e) implies (c). Suppose $s_i \to s$, $s(\gamma_i) = p(s_i)$ and $r(\gamma_i) \to p(s)$. Then we must have

$$(r(\gamma_i), [s_i]) \to (p(s), [s])$$

in $G^{(0)} * S/G$. Since ϕ is a homeomorphism,

$$\phi^{-1}(r(\gamma_i), [s_i]) = (r(\gamma_i), p(s_i)) \cdot s_i \to \phi^{-1}(p(s), [s]) = s.$$

However, since the action of R is the factorization of the action of G, we have $(r(\gamma_i), p(s_i)) \cdot s_i = \gamma_i \cdot s_i$ and the result follows.

Finally, let us show that (c) implies (a). Suppose $(u_i, v_i) \to (u, v)$ in R_P and $s_i \to s$ such that $p(s_i) = v_i$ and $p(s) = v$. Fix γ_i and γ in G such that $\pi(\gamma_i) = (u_i, v_i)$ for all i and $\pi(\gamma) = (u, v)$. Since the range map on G is open, we can pass to a subnet, relabel, and find η_i such that $r(\eta_i) = u_i$ and $\eta_i \to \gamma_i$. But then $s(\eta_i^{-1}\gamma_i) = p(s_i)$ for all i and

$$r(\eta_i^{-1}\gamma_i) = s(\eta_i) \to s(\gamma) = p(s).$$

It follows from part (c) that $\eta_i^{-1}\gamma_i \cdot s \to s$. Since $\eta_i \to \gamma$ we must have $\gamma_i \cdot s \to \gamma \cdot s$ and that the action of R_P is continuous.

The proof that (b), (d) and (f) are equivalent is almost exactly the same and we will not reproduce it here. We will end by showing that (c) and (d) are equivalent. Suppose (c) holds and that ω_i, ω and γ are as in the statement of (d). Then given $s_i \to s$ such that $p(s_i) = r(\gamma_i)$ for all i and $p(s) = \hat{p}(\omega)$ we have

$$\gamma_i \cdot \omega_i(s_i) = \omega_i(\gamma_i^{-1} \cdot s_i).$$

Now $s(\gamma_i^{-1}) = p(s_i)$ and $r(\gamma_i^{-1}) = \hat{p}(\omega_i) \to \hat{p}(\omega) = p(s)$ so that we may apply part (c) to conclude that $\gamma_i^{-1} \cdot s \to s$. It follows that $\gamma_i \cdot \omega_i(s_i) \to \omega(s)$. Hence $\gamma_i \cdot \omega_i \to \omega$. Thus (c) implies (d). Now suppose (d) holds. We can replace S by $\widehat{S}$ in the above argument to conclude that given s_i, s and γ as in (c) then $\gamma_i \cdot \hat{s}_i \to \hat{s}$ in $\widehat{\widehat{S}}$. However, $s \mapsto \hat{s}$ is an isomorphism by Theorem 2.52 and it is easy to see that $\eta \cdot \hat{t} = (\eta \cdot t)^\wedge$ so that (c) follows. $\square$

Of course, this proposition allows us to relax the conditions of Proposition 7.35 in the obvious way. Furthermore, now that we have all of these equivalent conditions it is easy for us to see that there are some fairly wide classes of groupoids for which the action of R_P is continuous.

Definition 7.38. Suppose G is a locally compact Hausdorff groupoid which acts on a locally compact Hausdorff space X. A set A in X is *wandering* if

$$\{\gamma \in G : \gamma \cdot A \cap A \neq \emptyset\}$$

is compact. A locally compact Hausdorff groupoid G is *Cartan* if every point in the unit space of G has a neighborhood which is wandering.

Proposition 7.39. *Let G be a second countable, locally compact Hausdorff groupoid with a Haar system and continuously varying abelian stabilizers S. Then the action of R_P on S is continuous if any of the following are true.*

(a) *G is proper.*

(b) *G is transitive.*

(c) *G is Cartan.*

(d) *G is principal.*

(e) *G is the transformation groupoid of an abelian group action.*

Proof. We know that the topology on R_Q is finer than the topology on R_P. Suppose $\pi : G \to R_P$ is open. If $O \subset R_Q$ is open then $\pi(\pi^{-1}(O)) = O$ is open in R_P. Hence $R_P = R_Q$. Since the action of R_Q is always continuous by Proposition 7.34, it clearly suffices to show that π is open. If G is transitive then this follows from [MRW87, Theorem 2.2B]. Suppose G is proper and that we have $\gamma_i \in G$ and $\gamma \in G$ such that $\pi(\gamma_i) \to \pi(\gamma)$. Then in particular $s(\gamma_i) \to s(\gamma)$ and $\gamma_i \cdot s(\gamma_i) \to \gamma \cdot r(\gamma)$. Since the action of G on its unit space is proper, by definition, we can then pass to a subnet, relabel, and find $\eta \in G$ so that $\gamma_i \to \eta$. It follows quickly that $\pi(\gamma) = \pi(\eta)$ and therefore $\gamma\eta^{-1} \in S$. Using the fact that p is open we can pass to another subnet, relabel, and find $s_i \in S$ such that $s_i = r(\gamma_i)$ for all i and $s_i \to \gamma\eta^{-1}$. But then $\pi(s_i\gamma_i) = \pi(\gamma_i)$ for all i and $s_i\gamma_i \to \gamma$. It follows that π is open.

Moving on, suppose G is principal. Then the action of R_P on $S = G^{(0)}$ is trivially continuous. Next, suppose G is Cartan and we have $\gamma_i \in G$, $s_i \in S$ and $s \in S$ such that $s_i \to s$, $s(\gamma_i) = p(s_i)$ and $r(\gamma_i) \to p(s)$. Let W be a wandering neighborhood of $p(s)$. Then $s(\gamma_i) \to p(s)$ so that $s(\gamma_i)$ is eventually in W. However $r(\gamma_i) \to p(s)$ as well so that eventually $r(\gamma_i) \in W$. But we can then pass to a subnet, relabel, and assume that $s(\gamma_i), r(\gamma_i) \in W$ for all i. This implies that

$$\gamma_i \in \{\gamma \in G : \gamma \cdot W \cap W \neq \emptyset\}$$

for all i. Since this set is compact we may pass to a subnet, relabel, and find $\gamma \in G$ such that $\gamma_i \to \gamma$. It follows quickly that $s(\gamma) = r(\gamma) = p(s)$ so that $\gamma \in S_{p(s)}$ and therefore

$$\gamma_i \cdot s_i \to \gamma \cdot s = s.$$

Finally, suppose $G = H \ltimes X$ where H is an abelian group which acts continuously on X. Suppose $(s_i, x_i) \in S$, $(s, x) \in S$ and $(t_i, y_i) \in G$ such that $s(t_i, y_i) = t_i^{-1} \cdot y_i = x_i$ for all i and $y_i \to x$. Observe that

$$(t_i, y_i) \cdot (s_i, x_i) = (t_i s_i t_i^{-1}, y_i) = (s_i, y_i).$$

Since $(s_i, y_i) \to (s, x)$ we are done. $\square$

At this point we would like to show that given a second countable locally compact Hausdorff groupoid G with a Haar system and abelian, continuously varying stabilizers S such that $G^{(0)}/G$ and $\widehat{S}/G$ are Hausdorff then the action of R_P on S is continuous. This would prove that if G has abelian, continuously varying stabilizers and $G^{(0)}/G$ is Hausdorff then $C^*(G)$ has Hausdorff spectrum if and only if the action of R_P on S is continuous. Unfortunately, as we demonstrate in the following examples, this is not true.

Example 7.40. Let Y be as in Example 7.32 and recall that $\mathbb{Z}$ acts on Y via the ϕ_n also defined there. Let $G = \mathbb{Q}_D \rtimes_\phi \mathbb{Z}$ be the semidirect product where $\mathbb{Q}_D$ denotes the rationals equipped with the discrete topology. Furthermore, we define

$$\phi(n)(r) = r2^n \tag{7.9}$$

for all $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$. It is clear that $\phi(r)$ is an automorphism of $\mathbb{Q}_D$. Furthermore, it is easy to show that ϕ is a homomorphism from $\mathbb{Z}$ into the automorphism group of $\mathbb{Q}_D$. Thus G is a locally compact Hausdorff group which is second countable because it is, in fact, countable. Let the second factor of G act on Y as in Example 7.32. In other words, let

$$(r, n) \cdot y = n \cdot y$$

where $n \cdot y$ is defined via the ϕ_n. As in Example 7.33 this gives us a continuous group action of G on Y. Thus the transformation group $G \ltimes Y$ is a second countable, locally compact Hausdorff groupoid with a Haar system. Again as in Example 7.33, we can see that the stabilizer subgroups are given by $S_x = \{(r, 0) : r \in \mathbb{Q}\}$ and that they must vary continuously since they are constant. Furthermore, it is easy to see that they are abelian so that $G \ltimes Y$ has continuously varying abelian stabilizers. It will be important for us to observe that S is isomorphic to $\mathbb{Q}_D \times X$ via the isomorphism

$$((q, 0), x) \mapsto (q, x).$$

In fact, we will often just drop the extra zero and confuse stabilizers of G with stabilizers of $G \ltimes Y$. Finally, $\{x_n\}_{n=0}^\infty$ forms a set of representatives for the orbit space and it is not difficult to show that Y/G is actually homeomorphic to $\{x_n\}_{n=0}^\infty$ and is therefore Hausdorff. Next, given $((q, n), y) \in G \ltimes Y$ and $(r, x) \in S$ we have

$$((q, n), y) \cdot (r, x) = (r2^n, y). \tag{7.10}$$

We would like to show that $S/G \ltimes Y$ is Hausdorff. Suppose $[s_i] \to [s]$ and $[s_i] \to [t]$. in $S/G \ltimes Y$. Since X/G is Hausdorff we can perform the usual trick to see that we must have $\tilde{p}([s]) = \tilde{p}([t]) = [u]$. In fact, we may as well assume that $p(s) = p(t) = u$.

In this case that means $s = (r, u)$ and $t = (q, u)$ for $r, q \in \mathbb{Q}$. Now, we can pass to subnets and lift, twice, choose new representatives, and find $\gamma_i \in G \ltimes Y$ so that $s_i \to s$ and $\gamma_i \cdot s_i \to t$. Suppose $s_i = (r_i, x_i)$ and $\gamma_i = ((p_i, n_i), y_i)$. Then it follows from (7.10) that $\gamma_i \cdot s_i = (r_i 2^{n_i}, y_i)$. Hence $r_i \to r$ and $r_i 2^{n_i} \to q$. However, we gave $\mathbb{Q}_D$ the discrete topology so that, eventually,

$$q = 2^{n_i} r_i = 2^{n_i} r.$$

Now, if $r = 0$ then we have $q = 0$ so that $s = t$. If $r \neq 0$ we know that eventually $n_i = n = \log_2(q/r)$. We may as well pass to a subnet and assume this is always true. But then $n_i \cdot x_i \to n \cdot x$. However, we also have $n_i \cdot x_i = \gamma_i \cdot x_i = y_i \to x$. Thus $n \cdot x = x$. But the action of $\mathbb{Z}$ is free so that we must have $n = 0$. Thus $\log_2(q/r) = 0$ and $q = r$. It follows that $s = t$ and that S/G is Hausdorff.

We have shown that $G \ltimes Y$ is a second countable, locally compact Hausdorff groupoid with a Haar system and continuously varying abelian stabilizers. Furthermore, both Y/G and $S/G \ltimes Y$ are Hausdorff. However, we will show that the action of R_P on S is not continuous. Consider $s_n = (1, (2^{-2n}, 0, 0))$ for all n. Then it is clear that $s_n \to s = (1, (0, 0, 0))$ in S. Let $\gamma_n = ((0, 2n + 1), (2^{-2n-1}, 0, 0))$. Then we compute that

$$s(\gamma_n) = (2^{-2n}, 0, 0), \quad \text{and} \quad r(\gamma_n) = (2^{-2n-1}, 0, 0).$$

Thus $s(\gamma_n) = p(s_n)$ and $r(\gamma_n) \to p(s)$. However,

$$\gamma_n \cdot s_n = (2^{2n+1}, (2^{-2n-1}, 0, 0))$$

and this sequence doesn't converge to anything. It follows from Proposition 7.37 that the action of R_P on S is not continuous.

This is not quite the example we are looking for. We also want to know that $\widehat{S}/G$ is Hausdorff so that the spectrum of $C^*(G)$ is as well. This may very well be true in Example 7.40, but in order to compute it we would need to work with the dual of the discrete rationals, which is ugly indeed. Furthermore, there doesn't seem to be any inherent reason why $\widehat{S}/G$ should be Hausdorff whenever S/G is. However, we can be tricky and form a "dualized" version of Example 7.40.

Example 7.41. Let $Y, \mathbb{Z}$ and ϕ_n be as in Example 7.32. Now let $H = \widehat{\mathbb{Q}_D} \rtimes_\psi \mathbb{Z}$ be the semidirect product of $\mathbb{Z}$ by the dual of $\mathbb{Q}_D$. We define

$$\psi(n)(\omega)(q) = \omega(\phi(-n)(q)) = \omega(2^{-n} q)$$

for $n \in \mathbb{Z}$, $\omega \in \widehat{\mathbb{Q}_D}$ and $q \in \mathbb{Q}$. Recall that, since $\mathbb{Q}_D$ is discrete, the topology on $\widehat{\mathbb{Q}_D}$

is just the topology of pointwise convergence. If $\omega_i \to \omega$ in $\widehat{\mathbb{Q}_D}$ then

$$\psi(n)(\omega_i)(q) = \omega_i(2^{-n}q) \to \omega(2^{-n}q) = \psi(n)(\omega)(q).$$

Thus $\psi(n)$ is a continuous function. Furthermore, $\psi(0)$ is the identity and

$$\psi(n+m)(\omega)(q) = \omega(2^{-m}2^{-n}q) = \psi(n)(\psi(m)(\omega))(q).$$

Thus $\psi(n+m) = \psi(n)\circ\psi(m)$. It now follows that each $\psi(n)$ is an automorphism of $\widehat{\mathbb{Q}_D}$ and that ϕ is a homomorphism. Thus the semidirect product is well defined and H is a locally compact Hausdorff group. Furthermore, the topology on $\widehat{\mathbb{Q}}_D$ is the smallest topology such that all the point evaluations ev_q are continuous. In particular, if we fix a countable basis $\{U_i\}$ of $\mathbb{T}$ then the collection $\{\mathrm{ev}_q^{-1}(U_i)\}$ ranging over all $q \in \mathbb{Q}$ and i forms a countable sub-basis for the topology. Hence $\widehat{\mathbb{Q}_D}$ is second countable and therefore H is second countable. Just as in Example 7.40 let the second factor of H act on Y so that $(\omega, n)\cdot y = n\cdot y$. Then, as usual, given $y \in Y$ the stabilizer subgroup is $T_y = \{(\omega, 0) \in H : \omega \in \widehat{\mathbb{Q}_D}\}$. In particular, T_y abelian and the stabilizers are continuously varying. In fact, we can identify the stabilizer subgroupoid T of $H \ltimes Y$ with $\widehat{\mathbb{Q}_D} \times Y$. Thus the dual bundle is, using Pontryagin duality, $\widehat{T} = \mathbb{Q}_D \times Y$. It's no accident that $\widehat{T}$ is isomorphic to the bundle S from Example 7.40. We are going to show that the action of $H \ltimes Y$ on $\widehat{T}$ is nearly the same as the action of $G \ltimes Y$ on S. Given $(q, x) \in \widehat{T}$ and $((\omega, n), y) \in H \ltimes Y$ such that $-n\cdot y = x$ we have

$$\begin{aligned}
((\omega, n), y)\cdot(q, x)(\rho, y) &= (q, x)(((\omega, n), y)^{-1}\cdot(\rho, y)) \\
&= (q, x)(\psi(-n)(\omega^{-1}\rho\omega), x) \\
&= \phi(-n)(\rho)(q) = \rho(2^n q) \\
&= (2^n q, y)(\rho, y)
\end{aligned}$$

This implies that

$$((\omega, n), y)\cdot(q, x) = (2^n q, y). \tag{7.11}$$

Observe that (7.10) implies that $[(q, x)] = [(r, y)]$ in $S/G \rtimes Y$ if and only if $y = n\cdot x$ and $q = 2^n r$ for some $n \in \mathbb{Z}$. However, (7.11) implies the same thing about $\widehat{T}/H \rtimes Y$. In particular both spaces have the same orbits so that the quotient spaces $S/G \ltimes Y$ and $\widehat{T}/H \ltimes Y$ are identical. What's more, we showed that $S/G \ltimes Y$ was Hausdorff in Example 7.40.

Thus we have demonstrated that $H \rtimes Y$ is a second countable, locally compact Hausdorff groupoid with a Haar system and continuously varying abelian stabilizer. Furthermore, the orbit space Y/H is still homeomorphic to $\{x_n\}$ and is Hausdorff.

We also showed that $\widehat{T}/H \ltimes Y$ is Hausdorff. We will now see that the action of R_P on $\widehat{T}$ is not continuous. We will do this by first observing that $G \rtimes Y$ and $H \rtimes Y$ both have unit space Y, they both act in the same way, and therefore the orbit groupoid R is the same in each case. Furthermore, since R_P inherits its topology from $Y \times Y$, it follows that R_P is the same for both $G \ltimes Y$ and $H \ltimes Y$. It now follows from (7.10) and (7.11) that R_P acts on S and $\widehat{T}$ in the same manner. Thus, since R_P does not act continuously on S, it does not act continuously on $\widehat{T}$. Furthermore Proposition 7.37 now implies that the action of R_P on T must not be continuous either.

At this point we are about done since we have constructed a groupoid with continuously varying abelian stabilizer such that $G^{(0)}/G$ and $\widehat{S}/G$ are Hausdorff, but the action of R_P isn't continuous. In particular, in this case the spectrum of $C^*(G)$ is Hausdorff. This shows that whatever condition is equivalent to assuming Hausdorff spectrum is weaker than requiring R_P to act continuously on the stabilizers.

Remark 7.42. One of the implications of all of this is that, as a hypothesis for groupoids, continuously varying abelian stabilizers does not play the same role as abelian does for groups. Furthermore, it is notable that Examples 7.40 and 7.41 are transformation *group* actions. So, in some sense, the notion of groupoids being "generalized transformation groups" holds true in that we didn't have to leave the transformation group setting to find counter examples.

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Notation and Symbol Index

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